Computer Graphics

Lecture 5: raycasting

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Where are we?

- We have seen how to represent geometry on a computer
  - Coordinate systems
  - Geometric entities
  - Geometric transformations
- How can we create an image from them?...

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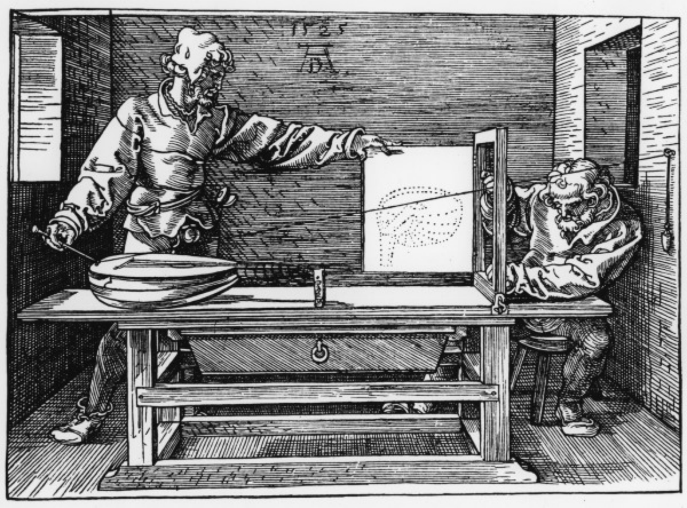
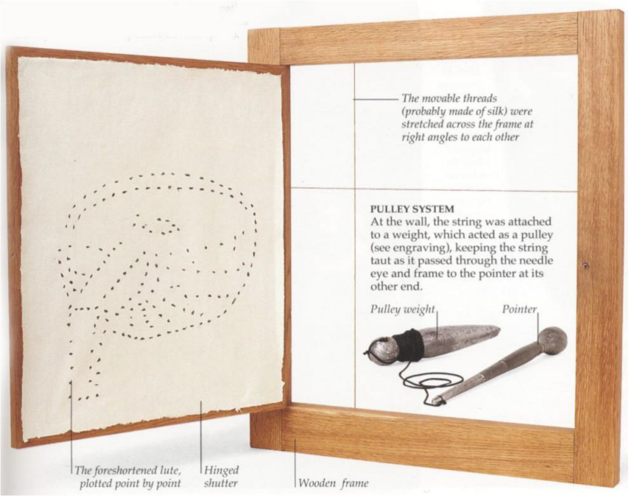
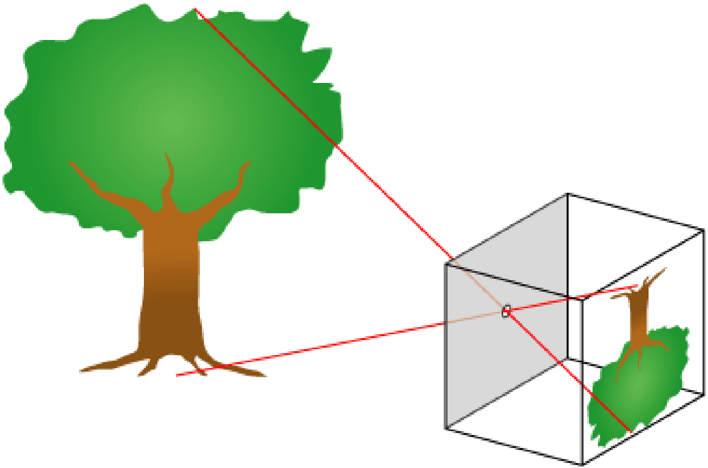
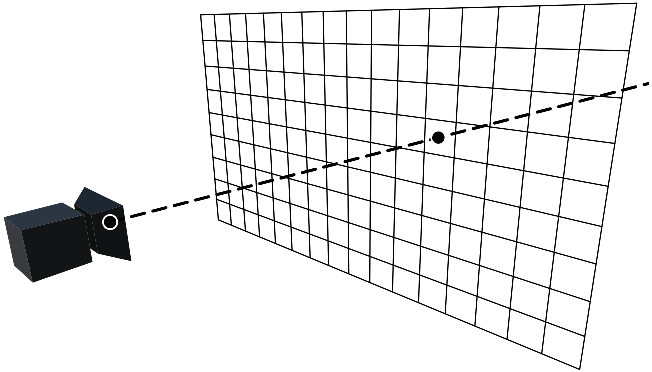


Albrecht Dürer, 1525


The movable threads (probably made of silk) were stretched across the frame at right angles to each other
PULLEY SYSTEM
At the wall, the string was attached to a weight, which acted as a pulley (see engraving), keeping the string taut as it passed through the needle eye and frame to the pointer at its other end.
Pulley weight
Pointer
The foreshortened lute, plotted point by point
Hinged shutter
Wooden frame





Motivation

- Think of each pixel as a small window to the world
- What color should the pixel be? $\rightarrow$ Let's see what the world looks like from there and assign a color to the pixel based on that!

Raycasting

```
For each pixel:
   Construct a ray from the eye

   For each object in the scene:
      Check if the ray intersects the object

   The color will be
   the closest intersected object's color
```

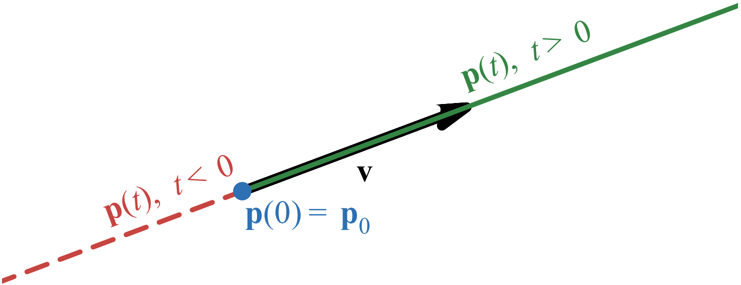
Ray

- The ray has
  - $\mathbf{p}_0$ origin
  - and $\mathbf{v}$ direction
- Parametric form:

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v},$$

  where $t > 0$ (half-line!).
- $t = 0$?, $t < 0$? origin of the ray, behind the ray

Ray



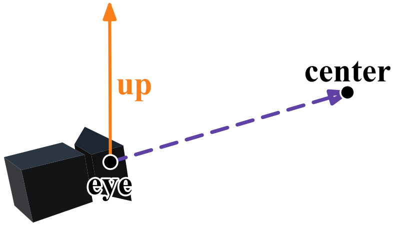
Question

- What will be the origin?
- What will be the direction?
- How do we intersect the ray with anything?

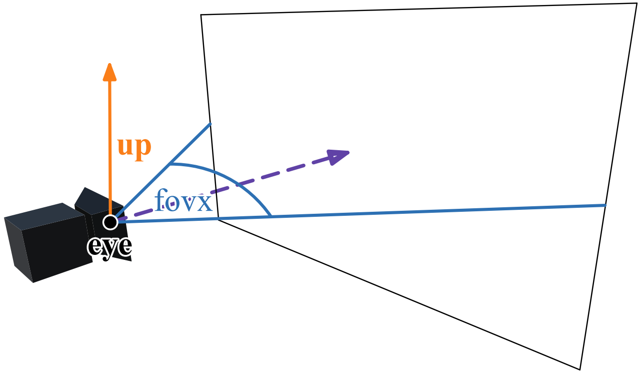
Generating rays

- From the eye position, we cast rays through the center of each pixel
- Now: we want to project centrally to an eye, with a rectangular part of the projection plane corresponding to the screen
- Eye/Camera properties:
  - eye position (**eye**),
  - a point to look at (**center**),
  - upward vector in the world (**up**),
  - opening angle, horizontal and vertical field of view (*fovx, fovy*).
  - (screen size. Now be given: $2\tan\left(\frac{fovx}{2}\right) \times 2\tan\left(\frac{fovy}{2}\right)$ size)
- We will use them to specify the world coordinates of the pixel (i, j)

Camera properties



Camera properties



Ray creation

Let us find the (right-handed) $\mathbf{u}, \mathbf{v}, \mathbf{w}$ coordinate system of the camera!

- Let the camera face towards $-Z$!

$$\mathbf{w} = \frac{\mathbf{eye} - \mathbf{center}}{|\mathbf{eye} - \mathbf{center}|}$$

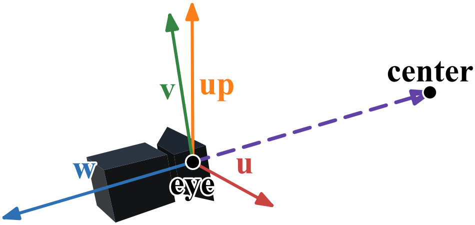
- Let the X axis be perpendicular to **w** and **up**!

$$\mathbf{u} = \frac{\mathbf{up} \times \mathbf{w}}{|\mathbf{up} \times \mathbf{w}|}$$

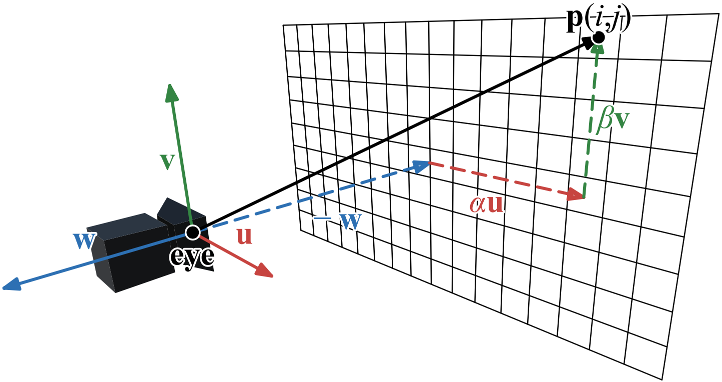
- Let Y axis be perpendicular to **u** and **w**:

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

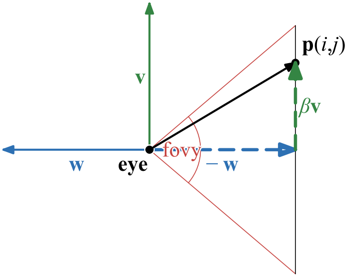
Camera coordinate system



Coordinates of the pixel at (i, j)



Coordinates of the pixel at (i, j)



- *width*, *height* is the size of the screen in pixels
- i, j are the pixel coordinates, where $(0, 0)$ is the top-left corner

- Let $\mathbf{p}$ be the center of i, j pixel and the projection plane is at a unit distance away from the point of view! Then

$$\mathbf{p}(i, j) = \mathbf{eye} + (\alpha \mathbf{u} + \beta \mathbf{v} - \mathbf{w}).$$

- Where

$$\alpha = \tan\left(\frac{fovx}{2}\right) \cdot \frac{i - width/2}{width/2},$$

$$\beta = \tan\left(\frac{fovy}{2}\right) \cdot \frac{height/2 - j}{height/2}.$$

The equation of the ray

- The ray is a half-line, which we can represent with its starting point and direction vector.
- Let $\mathbf{p}_0$ the origin of the ray and $\mathbf{v}$ its direction, then

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}, \quad t \geq 0$$

  gives all the points of the ray.
- The starting point of the rays is $\mathbf{p}_0 = \mathbf{eye}$
- And the direction vector of the ray can be calculated from the pixel's position as $\mathbf{v} = \frac{\mathbf{p}(i,j) - \mathbf{eye}}{|\mathbf{p}(i,j) - \mathbf{eye}|}$

Intersections

- Ray-tracing programs will perform intersections during the majority of their runtime
- Let's look at the intersections with simple geometric elements
- Our ray always has the previously seen $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$ form, where we will further assume that $|\mathbf{v}| = 1$
- Then the ray parameter t is exactly the distance between $\mathbf{p}(t)$ and $\mathbf{p}_0$!

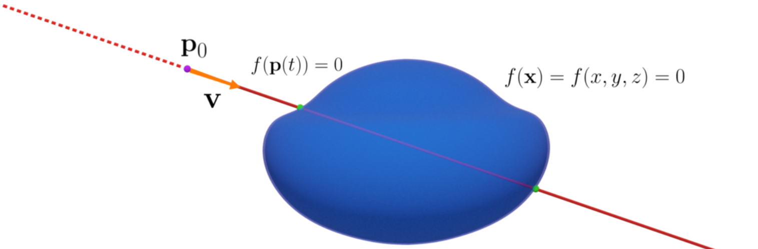
Intersections: parametric ray – implicit surface

- Let $f(\mathbf{x}) = f(x, y, z) = 0$ an implicit equation, which defines the surface we want to intersect ($\mathbf{x} \in \mathbb{R}^3$)
- The equation of our ray $\forall t \in [0, \infty)$ defines a point in space $\rightarrow$ Let us plug it into the implicit equation!
- We need to solve the equation for t:

$$f(\mathbf{p}(t)) = 0$$

- Based on the resulting t:
  - If $t > 0$, the ray intersects the object and it's in front of the camera
  - If $t = 0$ the origin of the ray is on the surface
  - If $t < 0$, the object is „behind" the ray and intersects it (but we want $t > 0$!)

Intersections: parametric ray – implicit surface



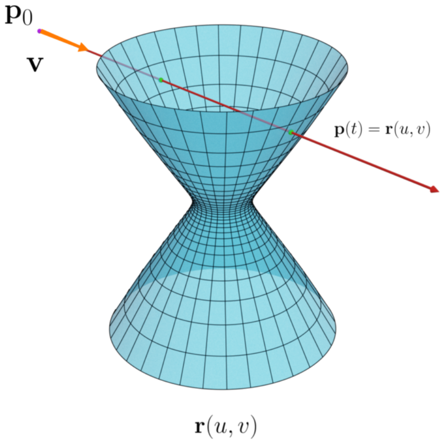
Intersections: parametric ray – parametric surface

- Let $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]^T$ be a parametric surface
- Need to find a ray parameter t for which there exist (u, v), such that

$$\mathbf{p}(t) = \mathbf{r}(u, v)$$

- This is a system of equations with three unknowns (t, u, v) and three equations (one for each x, y, z coordinate)
- t should be checked in the same way as before, but now we should also pay attention to (u, v), whether it is in the parameter range of our surface (e.g. $(u, v) \in [0, 1]^2$ needed)!

Intersections: parametric ray – parametric surface



Ray – transformed object intersection

- Let **M** the transformation matrix of the object.
- Task: Find the intersection of the ray **r** and the object transformed by **M**!
- Problem: How do we transform a sphere? Point by point? Should we rewrite the formula? ...
- Solution: Let's transform the ray instead!

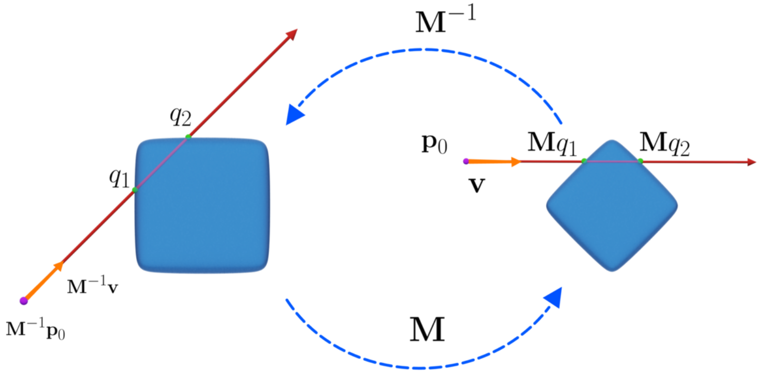
Ray – transformed object intersection

Theorem

The intersection of $\mathbf{r}$ ray and an object transformed by $\mathbf{M}$ transformation $\equiv$ the intersection of the $\mathbf{r}$ ray transformed by $\mathbf{M}^{-1}$ and the object.

- $\mathbf{M} \in \mathbb{R}^{4\times 4}$, homogeneous transformation
- Ray's origin: $\mathbf{p}_0 = (p_x, p_y, p_z) \to [p_x, p_y, p_z, 1]^T$
- Ray's direction: $\mathbf{v} = (v_x, v_y, v_z) \to [v_x, v_y, v_z, 0]^T$. Translation in $\mathbf{M}$ does not affect it.
- Transformed ray: $\widehat{\mathbf{r}}(t) = \mathbf{M}^{-1}\mathbf{p}_0 + t \cdot \mathbf{M}^{-1}\mathbf{v}$

Intersections: parametric ray – transformed object



Ray – transformed object intersection

- Intersection examination: use $\widehat{\mathbf{r}}(t)$!
- Intersection point: if on the original object (with the inverse transformed ray) $\mathbf{q}$, then on the transformed object $\mathbf{M} \cdot \mathbf{q}$.
- Distances must be recalculated in the original space!
- Normal vectors: $\mathbf{M}^{-T} \cdot \mathbf{n}$ instead of $\mathbf{n}$ (inverse-transpose) ($\mathbf{M} \in \mathbb{R}^{3\times 3}$).

Ray and implicit plane intersection

- Implicit plane: $Ax + By + Cz + D = 0$
- The ray

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

intersect the plane, if

$$A(p_x + tv_x) + B(p_y + tv_y) + C(p_z + tv_z) + D = 0$$

Ray and implicit plane intersection

- Solve it for t

$$t(Av_x + Bv_y + Cv_z) + Ap_x + Bp_y + Cp_z + D = 0$$
$$t = -\frac{Ap_x + Bp_y + Cp_z + D}{Av_x + Bv_y + Cv_z}$$

- Visible from our point of view, if $t > 0$

Ray and plane given by a point and a normal

- Let $\mathbf{q}_0$ be a point of the plane, $\mathbf{n}$ its normal,
- Let $\mathbf{p}_0$ a point of a line, $\mathbf{v}$ its direction.
- The equation of the line:

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$$

- The equation of the plane:

$$\langle \mathbf{n}, \mathbf{x} - \mathbf{q}_0 \rangle = 0$$

– every $\mathbf{x}$ point of the plane satisfies the equation.

Ray and plane given by a point and a normal

- Substitute $\mathbf{x}$ with $\mathbf{p}(t)$:

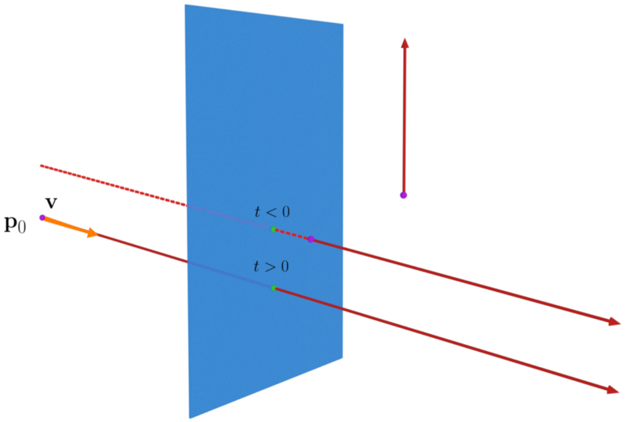
$$\langle \mathbf{n}, \mathbf{p}_0 + t\mathbf{v} - \mathbf{q}_0 \rangle = 0,$$

$$\langle \mathbf{n}, \mathbf{p}_0 \rangle + t\langle \mathbf{n}, \mathbf{v} \rangle - \langle \mathbf{n}, \mathbf{q}_0 \rangle = 0,$$

$$t = \frac{\langle \mathbf{n}, \mathbf{q}_0 \rangle - \langle \mathbf{n}, \mathbf{p}_0 \rangle}{\langle \mathbf{n}, \mathbf{v} \rangle} = \frac{\langle \mathbf{n}, \mathbf{q}_0 - \mathbf{p}_0 \rangle}{\langle \mathbf{n}, \mathbf{v} \rangle},$$

  if $\langle \mathbf{n}, \mathbf{v} \rangle \neq 0$.
- The ray intersects the plane if: $t > 0$.
- If $\langle \mathbf{n}, \mathbf{v} \rangle = 0$, the ray is parallel to the plane, and there is no intersection, or the ray runs along the plane

Intersections: ray – plane



Ray and parametric plane intersection

- The plane can be represented by one of its points, $\mathbf{q}_0$ and two spanning vectors $\mathbf{i}, \mathbf{j}$: $\mathbf{s}(u, v) = \mathbf{q}_0 + u\mathbf{i} + v\mathbf{j}$
- Intersection with $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$ ray: find t and u, v such that

$$\mathbf{p}(t) = \mathbf{s}(u, v)$$

- The equation is

$$\mathbf{p}_0 + t\mathbf{v} = \mathbf{q}_0 + u\mathbf{i} + v\mathbf{j}$$

- After rearranging it, we get

$$\mathbf{p}_0 - \mathbf{q}_0 = -t\mathbf{v} + u\mathbf{i} + v\mathbf{j}$$

Ray and parametric plane intersection

- This system of equations consists of three linear equations, which can be solved, if $\mathbf{v}, \mathbf{i}, \mathbf{j}$ are linearly independent
- Matrix form:

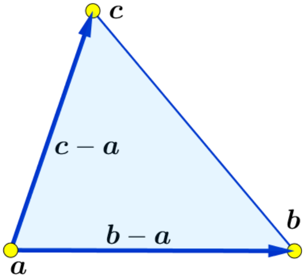
$$\begin{bmatrix} p_{0x} - q_{0x} \\ p_{0y} - q_{0y} \\ p_{0z} - q_{0z} \end{bmatrix} = \begin{bmatrix} -v_x & i_x & j_x \\ -v_y & i_y & j_y \\ -v_z & i_z & j_z \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix}$$

- We see the plane, if $t > 0$ (now the parameter range is $u, v \in \mathbb{R}$, so it will be satisfied)

Ray and triangle intersection I.

- A triangle is uniquely defined by its three vertices.
- If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the vertices of the triangle, then a parametric form of the associated plane

$$\mathbf{s}(u, v) = \mathbf{a} + u(\mathbf{b} - \mathbf{a}) + v(\mathbf{c} - \mathbf{a})$$



- With previous notations: $\mathbf{q}_0 = \mathbf{a}$, $\mathbf{i} = \mathbf{b} - \mathbf{a}$ and $\mathbf{j} = \mathbf{c} - \mathbf{a}$.

Ray and triangle intersection I.

- This is barycentric formulation too, because after we rearrange it, we get that

$$\mathbf{s}(u, v) = (1 - u - v)\mathbf{a} + u\mathbf{b} + v\mathbf{c},$$

  where the coefficients add up to 1.

- Having completed the intersection with the parametric plane, we obtain the barycentric coordinates of the point of intersection of the ray with the plane. Don't forget to check the ray parameter (t)!
- As a final step, we need to check if the intersection point is inside the triangle. This is true if and only if

$$0 \leq u,\ \ 0 \leq v \ \text{ and } \ 0 \leq 1 - u - v.$$

Ray and triangle intersection II.

- A triangle is uniquely defined by its three vertices.
- If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the vertices of the triangle, then the implicit pointnormal plane associated with the triangle is
  - its point is one of the $\mathbf{a}, \mathbf{b}, \mathbf{c}$ points
  - its normal
    $$\mathbf{n} = \frac{(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})}{\|(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})\|},$$
    where $\times$ is the cross product, and $\mathbf{n}$ is a unit vector.

Ray and triangle intersection II.

- First let's calculate the intersection point of the line and triangle's plane, let it be $\mathbf{p}$ (if it exists).
- Let $\lambda_1, \lambda_2, \lambda_3$ be the barycentric coordinates of point $\mathbf{p}$ relative to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that

$$\mathbf{p} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}$$

- $\mathbf{p}$ is inside of $\triangle$ if and only if

$$0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1.$$

Point on triangle

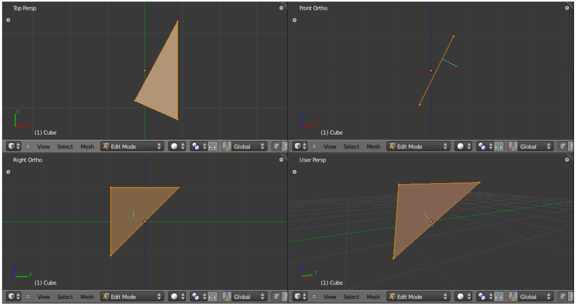
- We know that $\mathbf{p} = [x, y, z]^T = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}$. Then

$$
\begin{aligned}
x =& \lambda_1 a_x + \lambda_2 b_x + \lambda_3 c_x \\
y =& \lambda_1 a_y + \lambda_2 b_y + \lambda_3 c_y \\
z =& \lambda_1 a_z + \lambda_2 b_z + \lambda_3 c_z,
\end{aligned}
$$

  respectively $\lambda_1 + \lambda_2 + \lambda_3 = 1 \Rightarrow \lambda_3 = 1 - \lambda_1 - \lambda_2$
- For faster calculation, let's project the above onto a plane
- From the coordinate planes (XY, XZ or YZ), take the 2D projection of the triangle where the area of the triangle's projection is the largest! $\rightarrow$ where the normal of the triangle and plane are the "closest"
- For the projection, we simply omit the z, y, or x equation, respectively.

Point on triangle

We will choose the axis, along which the magnitude of the triangle's normal vector is the largest
(This way, it cannot happen that the triangle is perpendicular to the plane, and only one segment remains!)



Point on triangle

- e.g. Let z be the chosen axis. Then

$$x = \lambda_1 a_x + \lambda_2 b_x + \lambda_3 c_x$$
$$y = \lambda_1 a_y + \lambda_2 b_y + \lambda_3 c_y$$

- With $\lambda_3 = 1 - \lambda_1 - \lambda_2$, and after rearrangement:

$$x = \lambda_1 (a_x - c_x) + \lambda_2 (b_x - c_x) + c_x$$
$$y = \lambda_1 (a_y - c_y) + \lambda_2 (b_y - c_y) + c_y$$

Point on triangle

- Arranged for λ_1, λ_2 we get:

$$\lambda_1 = \frac{(b_y - c_y)(x - c_x) - (b_x - c_x)(y - c_y)}{(a_x - c_x)(b_y - c_y) - (b_x - c_x)(a_y - c_y)}$$
$$\lambda_2 = \frac{-(a_y - c_y)(x - c_x) + (a_x - c_x)(y - c_y)}{(a_x - c_x)(b_y - c_y) - (b_x - c_x)(a_y - c_y)}$$

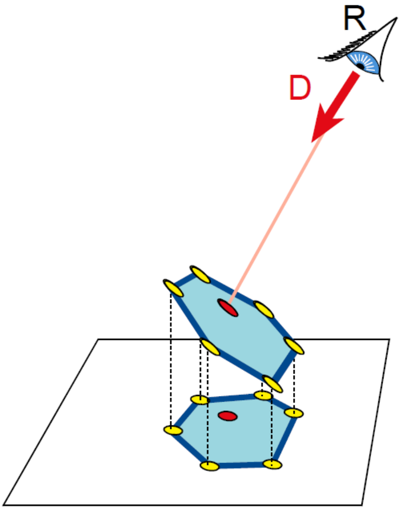
- The denominator can only be zero in the case of a degenerate triangle.
- **p** is inside the triangle if and only if

$$0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1.$$

Ray intersection with polygon

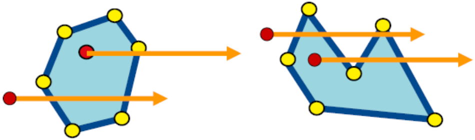
- Let's assume that the vertices of our polygon are in the same plane, then the intersection in two steps
  - Intersect the ray with the plane of the polygon
  - Decide whether the point of intersection is inside the polygon
- The second one should be done in a plane (either in the plane of the polygon, or in the plane of the projected polygon onto a coordinate plane)

Ray intersection with polygon



Point-in-polygon problem

- A point is inside the polygon if an arbitrary half-line (ray) starting from that point has an odd number of intersections with the edges of the polygon
- It also works for concave and star-shaped polygons



Ray–line segment intersection

- The parametric form of the segment between vertices $\mathbf{d}_i = [x_i, y_i]^T, \mathbf{d}_{i+1} = [x_{i+1}, y_{i+1}]^T$ of the polygon: $\mathbf{d}_{i,i+1}(s) = (1-s)\mathbf{d}_i + s\mathbf{d}_{i+1} = \mathbf{d}_i + s(\mathbf{d}_{i+1} - \mathbf{d}_i), s \in [0,1]$
- Let us intersect it with $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$ ray
- Now: the point $\mathbf{p}_0 = (x_0, y_0)$ is the point about which we want to decide whether it is inside the polygon, $\mathbf{v}$ is arbitrary
- Let $\mathbf{v} = (1, 0)$!
- We only need to solve $\mathbf{p}(t) = \mathbf{d}_{i,i+1}(s)$ equation on the y coordinate

Ray–line segment intersection

- Let's find where the line of side $\mathbf{d}_{i,i+1}(s)$ intersects the ray (=at which s is $d_{i,i+1}(s)_y = y_0$?)
- That is $y_0 = y_i + s(y_{i+1} - y_i)$
- solve for s: $s = \frac{y_0 - y_i}{y_{i+1} - y_i}$
- From this we get that x coordinate, where the ray intersects the segment by substituting it into $\mathbf{d}_{i,i+1}(s)$
- If $s \notin [0, 1]$: then the ray does not intersect the segment (only its line)
- If $t \leq 0$: the ray is inside of the segment or the segment is behind the ray

Ray – sphere intersection

- The $\mathbf{c} = (c_x, c_y, c_z)$ centered, r radius sphere's implicit equation is:

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

- With dot product:

$$\langle \mathbf{x} - \mathbf{c}, \mathbf{x} - \mathbf{c} \rangle - r^2 = 0,$$

where $\mathbf{x} = [x, y, z]^T$.

Ray – sphere intersection

- Let $\mathbf{p}_0$ a point on the line, $\mathbf{v}$ its direction vector.
- The equation of the line:

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$$

- By substituting it into the sphere's equation:

$$\langle \mathbf{p}_0 + t\mathbf{v} - \mathbf{c}, \mathbf{p}_0 + t\mathbf{v} - \mathbf{c} \rangle - r^2 = 0$$

- Expanded:

$$\langle t\mathbf{v} + (\mathbf{p}_0 - \mathbf{c}), t\mathbf{v} + (\mathbf{p}_0 - \mathbf{c}) \rangle - r^2 = 0$$

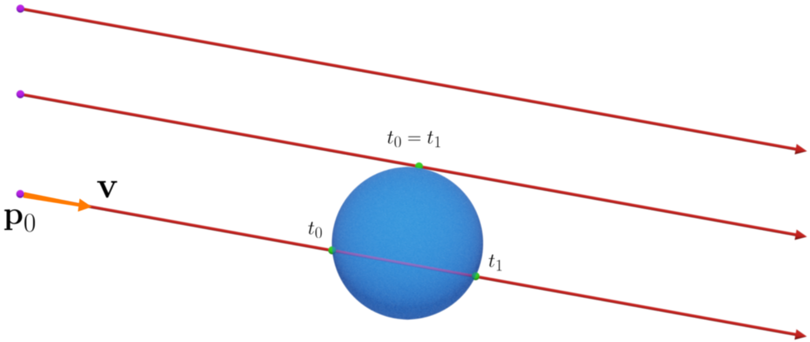
$$t^2 \langle \mathbf{v}, \mathbf{v} \rangle + 2t\langle \mathbf{v}, \mathbf{p}_0 - \mathbf{c} \rangle + \langle \mathbf{p}_0 - \mathbf{c}, \mathbf{p}_0 - \mathbf{c} \rangle - r^2 = 0$$

Ray – sphere intersection

$$t^2\langle \mathbf{v}, \mathbf{v}\rangle + 2t\langle \mathbf{v}, \mathbf{p}_0 - \mathbf{c}\rangle + \langle \mathbf{p}_0 - \mathbf{c}, \mathbf{p}_0 - \mathbf{c}\rangle - r^2 = 0$$

- This is a quadratic equation for t (we know everything else).
- Let $D = (2\langle \mathbf{v}, \mathbf{p}_0 - \mathbf{c}\rangle)^2 - 4\langle \mathbf{v}, \mathbf{v}\rangle(\langle \mathbf{p}_0 - \mathbf{c}, \mathbf{p}_0 - \mathbf{c}\rangle - r^2)$
- If $D > 0$: two solutions, the line intersects the sphere.
- If $D = 0$: one solution, the line is tangential.
- If $D < 0$: there is no real solution, no intersection.
- The ray parameter must then be checked $(t > 0)$: the smallest positive t is the solution (if it exists).

Ray – sphere intersection



Solving $ax^2 + bx + c = 0$

- In theory if $a \neq 0$ we can get the solution:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In practice when $a \approx 0$ it's a problem
  - With modifications we get that

$$x_{1,2} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

  we can get the two roots in this form ("citardauq" formula)

Solving $ax^2 + bx + c = 0$

- In theory if $a \neq 0$ we can get the solution:

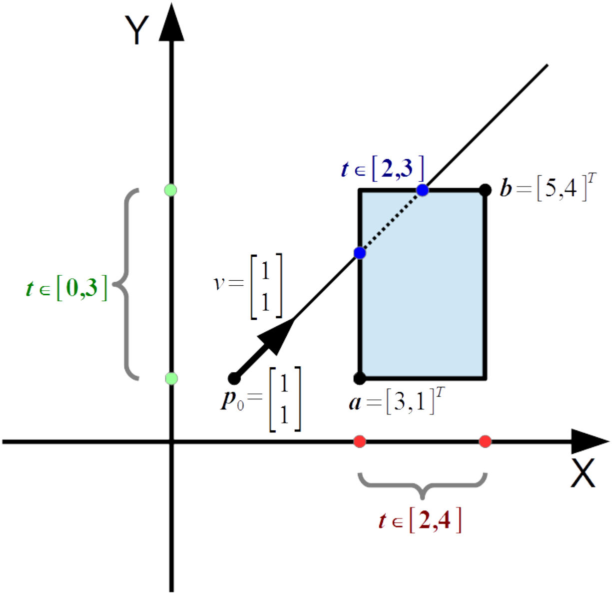
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In practice when $b >> 4ac$ it's a problem
  - Then $b^2 - 4ac \approx b^2$ that is, depending on the sign of b, either $-b + \sqrt{b^2 - 4ac}$ or $-b - \sqrt{b^2 - 4ac}$ we lose valuable decimals
  - Calculate one root on the branch where we don't subtract two nearly identical numbers from each other, we get the other root from the Vieta's formulas
  - That is, for example, if $b > 0$, then $x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{c}{ax_1}$

Ray – AAB intersection

- AAB = *Axis Aligned Box*, the sides of the rectangle (in 3D: faces of the box) are parallel to the coordinate lines / planes
- Let $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$ be a ray where $\mathbf{p}_0 = [x_0, y_0, z_0]^T, \mathbf{v} = [v_x, v_y, v_z]^T$ and let us represent the AAB by the endpoints of its diagonal, $\mathbf{a}$ and $\mathbf{b}$ ($\mathbf{a} < \mathbf{b}$)!
- Idea: compute a separate interval for t regarding each of the coordinates. The ray intersects the box if the intersection of the intervals is not empty.

Ray – AAB intersection



Ray – AAB intersection

- Let $t_{near} := -\infty, t_{far} := +\infty$
- If $v_x = 0$: perpendicular to x axis (e.g. vertical in 2D). Then there is no intersection if $x_0 \notin [a_x, b_x]$, otherwise we can skip x-coordinate calculations
- If $v_x \neq 0$, then let $t_1 := \frac{a_x - x_0}{v_x}, t_2 := \frac{b_x - x_0}{v_x}$
- If $t_1 > t_2$: swap t_1 and t_2!
- If $t_{near} < t_1$: $t_{near} := t_1$
- If $t_{far} > t_2$: $t_{far} := t_2$
- We do the above for the y and z coordinates as well

Ray – AAB intersection

- If $t_{near} > t_{far}$: the ray misses the box
- If $t_{far} < 0$: the box is behind the ray
- If $t_{near} < 0 < t_{far}$: the ray origin is in the box
- Else the intersections with the box will be at t_{near} and t_{far} (closer and further intersection points respectively)