

Computer Graphics

Lecture 4: curves and surfaces

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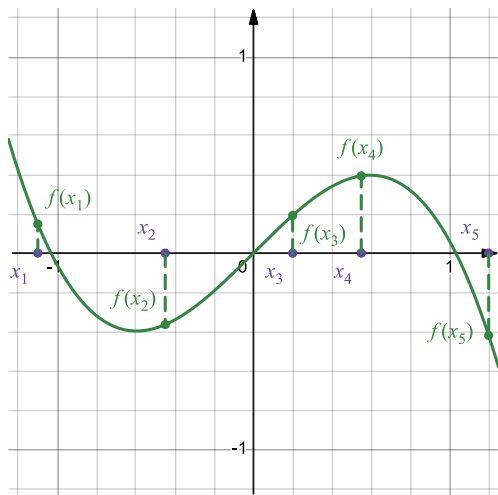
Motivation

- ▶ We can now represent the points in different coordinate systems
- ▶ How can we describe simple things, such as a line or a plane?
- ▶ We primarily examine the above question in the Cartesian coordinate system

Curves and surfaces

- ▶ We can represent curves and surfaces (which include lines and planes too) as a set of points.
- ▶ How can we define these sets?
 - ▶ explicit: $y = f(x), x \in \mathbb{R} \rightarrow \{(x, f(x)) \mid x \in \mathcal{D}_f \subset \mathbb{R}\}$
 - ▶ what if we want to „reverse“ it?
 - ▶ parametric: $\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in \mathbb{R}$
 - $\rightarrow \{\mathbf{p}(t) \mid t \in \mathcal{D}_\mathbf{p} \subset \mathbb{R}\}$
 - ▶ implicit: $f(x, y) = 0, (x, y) \in \mathbb{R}^2$
 - $\rightarrow \{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^2 \mid f(\mathbf{x}) = 0\}$

Explicit curves



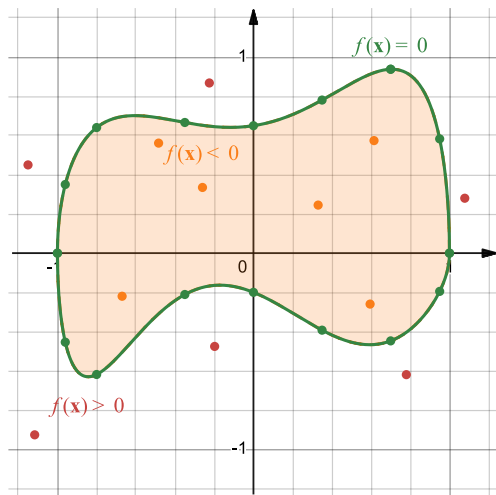
$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$y = f(x), x \in \mathbb{R}$$

Set of points:

$$\{(x, f(x)) \mid x \in \mathcal{D}_f \subset \mathbb{R}\}$$

Implicit curves



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

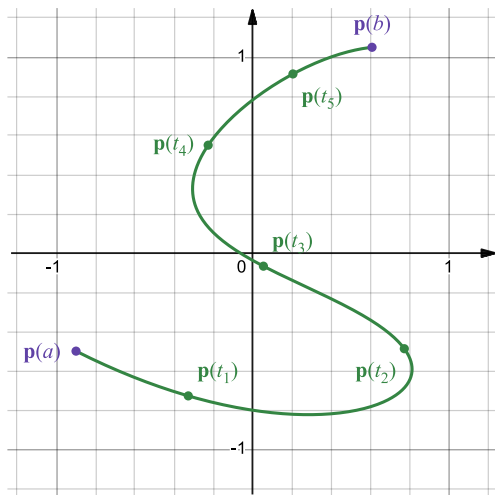
$$\mathbf{x} = (x, y) \in \mathbb{R}^2$$

$$f(x, y) = 0$$

Set of points:

$$\{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^2 \mid f(\mathbf{x}) = 0\}$$

Parametric curves



$$\mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}^2$$

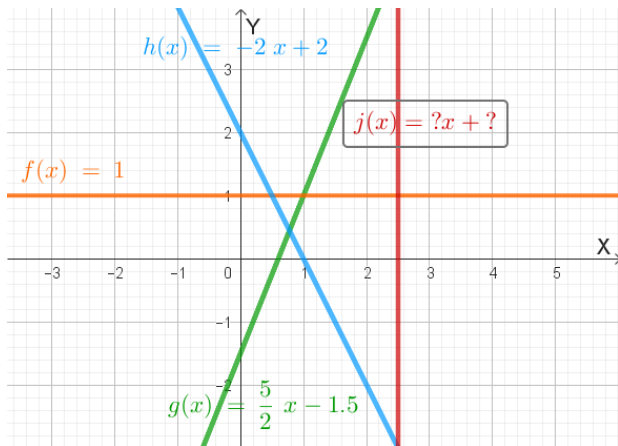
$$\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in \mathbb{R}$$

Set of points:

$$\{\mathbf{p}(t) \mid t \in \mathcal{D}_{\mathbf{p}} \subset \mathbb{R}\}$$

Line equation

- ▶ In high school: $y = mx + b$
- ▶ Problem: what about the vertical lines?



Normal vector equation of the line on the plane

- ▶ Let $\mathbf{p}(p_x, p_y)$ be a point on the line and $\mathbf{n} = [n_x, n_y]^T \neq \mathbf{0}$ vector, a **normal** perpendicular to the direction of the line:
- ▶ All $\mathbf{x}(x, y)$ points on the line satisfy

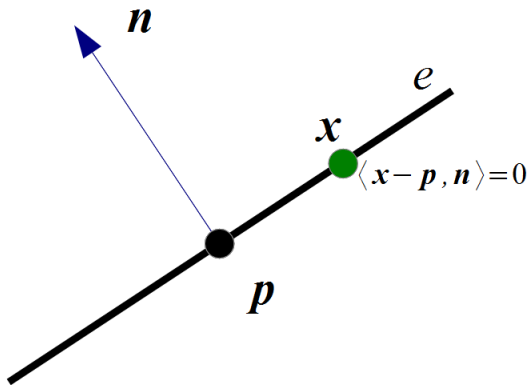
$$\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

$$(x - p_x)n_x + (y - p_y)n_y = 0$$

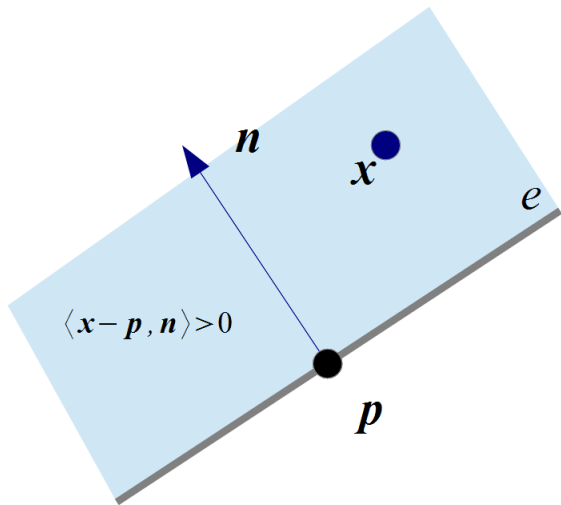
equation.

- ▶ $\langle \mathbf{x}' - \mathbf{p}, \mathbf{n} \rangle < 0$ and $\langle \mathbf{x}' - \mathbf{p}, \mathbf{n} \rangle > 0$ represents points on the two half-planes defined by our line.

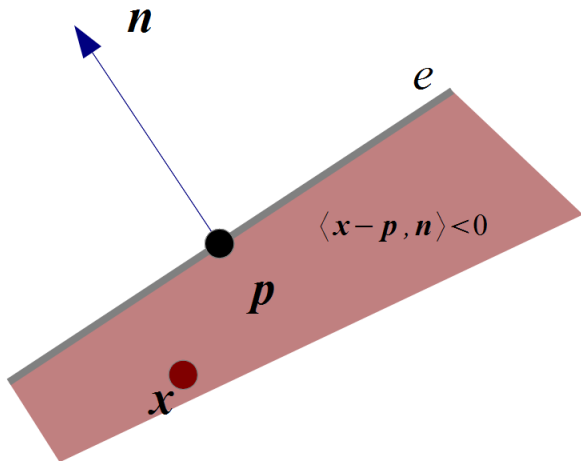
The two half-planes defined by line



The two half-planes defined by line



The two half-planes defined by line



The homogeneous implicit equation of the line on the plane

- ▶ The equation $ax + by + c = 0$ is the implicit equation of the line on the plane.
- ▶ In the previous representation, choosing $a = n_x$, $b = n_y$ and $c = -(p_x n_x + p_y n_y)$, $a^2 + b^2 \neq 0$ we get the implicit equation of the line going through \mathbf{p} , with \mathbf{n} normal
- ▶ If $a^2 + b^2 = 1$, then this is the *Hesse normal form* – in this case the normal vector has a unit length

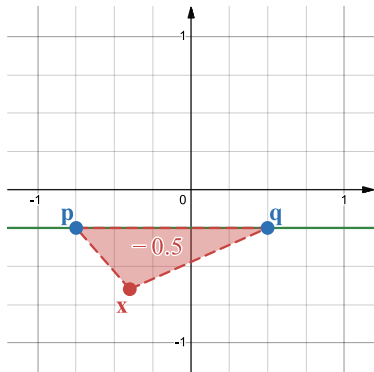
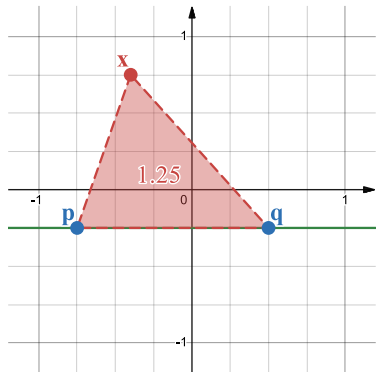
Homogeneous implicit equation with determinant

- ▶ Let $\mathbf{p}(p_x, p_y)$ and $\mathbf{q}(q_x, q_y)$ be two distinct points on the line then $\mathbf{x}(x, y)$ point belongs to the line if $\Delta(\mathbf{x}, \mathbf{p}, \mathbf{q}) = 0$, i.e.

$$\begin{vmatrix} x & y & 1 \\ p_x & p_y & 1 \\ q_x & q_y & 1 \end{vmatrix} = 0$$

- ▶ Remark: the above determinant is twice the signed area of the triangle spanned by $\mathbf{x}(x, y)$, $\mathbf{p}(p_x, p_y)$, $\mathbf{q}(q_x, q_y)$, which is equal to 0 \iff the points are in one line

Homogeneous implicit equation with determinant – figures



Parametric equation of lines – with direction vector (2D, 3D)

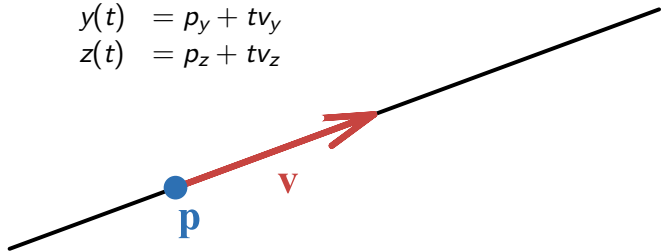
- Let $\mathbf{p}(p_x, p_y, p_z)$ be a point on the line and $\mathbf{v} = [v_x, v_y, v_z]^T \neq \mathbf{0}$ a direction vector of the line (a vector parallel to the line)

$$\mathbf{x}(t) = \mathbf{p} + t\mathbf{v} = \begin{bmatrix} p_x + tv_x \\ p_y + tv_y \\ p_z + tv_z \end{bmatrix} \rightarrow$$

$$x(t) = p_x + tv_x$$

$$y(t) = p_y + tv_y$$

$$z(t) = p_z + tv_z$$



Parametric equation of lines – with two points (2D, 3D)

- ▶ Then \mathbf{p} and \mathbf{q} are the points of the line. We get the previous case by using $\mathbf{v} = \mathbf{q} - \mathbf{p}$:

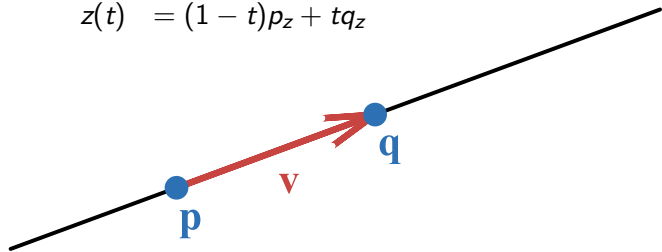
$$\mathbf{x}(t) = \mathbf{p} + t\mathbf{v} \rightarrow$$

$$\mathbf{x}(t) = (1 - t)\mathbf{p} + t\mathbf{q} \rightarrow$$

$$x(t) = (1 - t)p_x + tq_x$$

$$y(t) = (1 - t)p_y + tq_y$$

$$z(t) = (1 - t)p_z + tq_z$$



*Homogeneous coordinate form

- ▶ We can represent a line of the extended (projective) plane with a real number triplet $\mathbf{e} = [e_1, e_2, e_3]$, so-called *line coordinate*, using which for every point $\mathbf{x} = [x_1, x_2, x_3]^T$ of the line

$$\mathbf{e}\mathbf{x} = e_1x_1 + e_2x_2 + e_3x_3 = 0$$

- ▶ The line coordinate of the ideal line including every ideal point $[x_1, x_2, 0]$ of the plane is $[0, 0, 1]$.

*Polar coordinate form

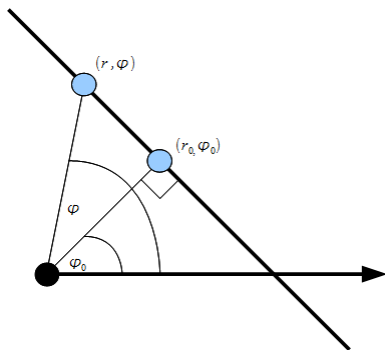
- ▶ The (implicit) equation of the half-line starting from the origin and making θ angle with the polar axis, in polar coordinate system:

$$\phi = \theta$$

- ▶ If our line does not pass through the origin, then let (r_0, ϕ_0) be the intersection of our line and a perpendicular line passing through the origin. Then, from the polar coordinates of our line, the radius can be written as a function of the polar angle in the following form:

$$r(\phi) = \frac{r_0}{\cos(\phi - \phi_0)}$$

*Polar coordinate form

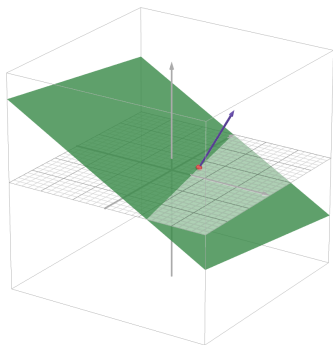


Normal vector equation of the plane

- ▶ Let $\mathbf{p}(p_x, p_y, p_z)$ a point on the plane and $\mathbf{n} = [n_x, n_y, n_z]^T$ normal vector orthogonal to the plane, then for every point \mathbf{x} of the plane:

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

- ▶ Half-spaces: $\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle < 0$,
 $\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle > 0$



Homogeneous, implicit equation of the plane

- ▶ Implicit form of the plane $ax + by + cz + d = 0$
- ▶ From this with $a = n_x$, $b = n_y$, $c = n_z$ and $d = -n_x p_x - n_y p_y - n_z p_z$ we get the equation for the plane going through \mathbf{p} point, \mathbf{n} normal vector
- ▶ Hesse normal form here too, if $a^2 + b^2 + c^2 = 1$

Homogeneous implicit equation with determinant

- ▶ We can write the equation as a determinant, where it's zero for every point (x, y, z) of the plane spanned by $\mathbf{p}(p_x, p_y, p_z)$, $\mathbf{q}(q_x, q_y, q_z)$, $\mathbf{r}(r_x, r_y, r_z)$ points

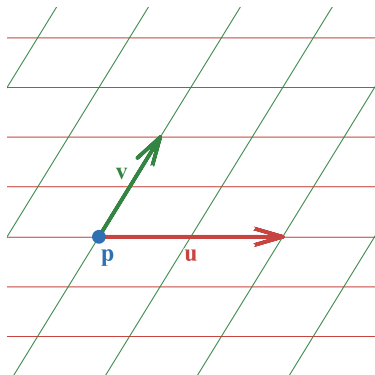
$$\begin{vmatrix} x & y & z & 1 \\ p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \\ r_x & r_y & r_z & 1 \end{vmatrix} = 0$$

Parametric equation of the plane – using spanning vectors

- ▶ Let \mathbf{p} point on the plane and \mathbf{u}, \mathbf{v} spanning vectors of the plane (basis vectors):

$$\mathbf{x}(s, t) = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where $s, t \in \mathbb{R}$.



Parametric equation of the plane – using three points

- ▶ Three points \mathbf{p} , \mathbf{q} , \mathbf{r} not falling into one line define a plane, then we can get every finite point \mathbf{x} of the plane

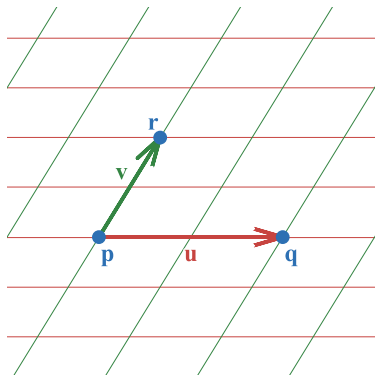
$$\mathbf{x}(s, t) = \mathbf{p} + s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p})$$

where $s, t \in \mathbb{R}$.

- ▶ We can get this from the former as well $\mathbf{u} = \mathbf{q} - \mathbf{p}$, $\mathbf{v} = \mathbf{r} - \mathbf{p}$
- ▶ This is a barycentric solution:

$$\mathbf{x}(s, t) = (1 - s - t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}$$

since $(1 - s - t) + s + t = 1$



*Homogeneous coordinate form

- ▶ A plane in projective space can be represented with 4-tuple $\mathbf{s} = [s_1, s_2, s_3, s_4]$ a "plane-coordinate", which for every point $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$ of the plane

$$\mathbf{s}\mathbf{x} = s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4 = 0$$

*Notable planes in homogeneous form

- ▶ $[0, 0, 0, c]$ ideal plane
- ▶ $[c, 0, 0, 0]$ the YZ plane
- ▶ $[0, c, 0, 0]$ the XZ plane
- ▶ $[0, 0, c, 0]$ the XY plane

Description of curves (2D)

- ▶ We can represent curves and surfaces (including lines and planes) as a set of points.
- ▶ How can we define these sets?
 - ▶ explicit: $y = f(x) \rightarrow \{(x, f(x)) \mid x \in \mathcal{D}_f \subset \mathbb{R}\}$
 - ▶ parametric: $\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \rightarrow \{\mathbf{p}(t) \mid t \in \mathcal{D}_\mathbf{p} \subset \mathbb{R}\}$
 - ▶ implicit: $f(x, y) = 0 \rightarrow \{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^2 \mid f(\mathbf{x}) = 0\}$
- ▶ But how do we draw them?
 - ▶ explicit & parametric: generate several points on the curve...
 - ▶ implicit: check pixels on screen if they are on the curve...

Transforming curves

How do we transform curves given in different representations?

▶ Explicit

▶ Vertical translation and scaling: modifying the function value

$$\rightarrow y = a \cdot f(x) + b$$

▶ Horizontal translation and scaling: modifying the parameter
(in the opposite direction)

$$\rightarrow y = f\left(\frac{x}{c} - d\right)$$

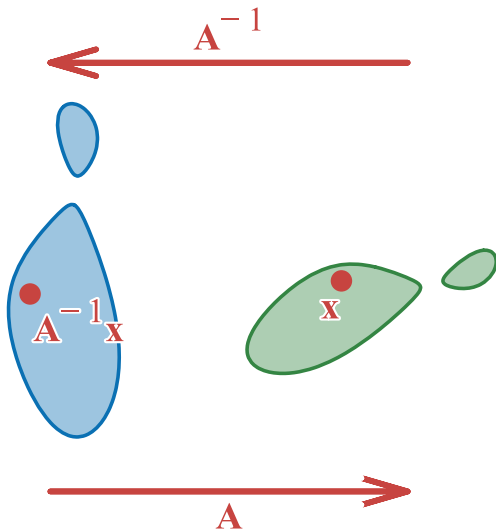
▶ Parametric: transforming the function value

$$\rightarrow \mathbf{A} \cdot \mathbf{p}(t)$$

▶ Implicit: transforming the parameter with the inverse

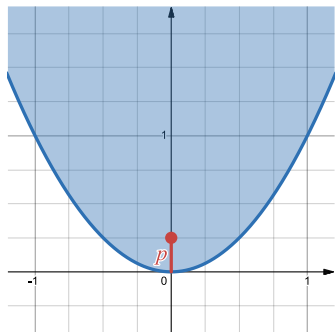
$$\rightarrow f(\mathbf{A}^{-1} \cdot \mathbf{x}) = 0$$

Transforming implicit curves



Parabola

- ▶ The parabola of focus point $(0, p)$ about y axis and crossing the origin can be written as
 - ▶ Implicit: $x^2 = 4py$
 - ▶ Explicit: $y = \frac{x^2}{4p}$, $x \in \mathbb{R}$
 - ▶ Parametric: $\mathbf{p}(t) = \begin{bmatrix} t \\ \frac{t^2}{4p} \end{bmatrix}$, $t \in \mathbb{R}$

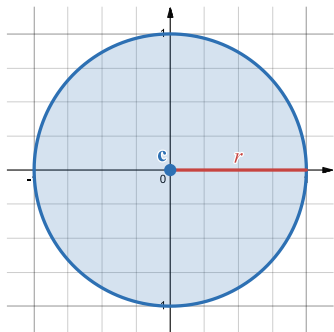


Parabola

- ▶ What if we want to translate the parabola from origin into \mathbf{c} point?
- ▶ In implicit and explicit formulation one has to work the coordinates of the translation (c_x, c_y) into the formulation (e.g. from the implicit we get $(x - c_x)^2 = 4p(y - c_y)$)
- ▶ In parametric form it is simply $\mathbf{p}(t) + \mathbf{c}$.

Circle

- ▶ A circle with $\mathbf{c} \in \mathbb{E}^2$ origin and r radius
 - ▶ Implicit form:
 $(x - c_x)^2 + (y - c_y)^2 = r^2$
 - ▶ Explicit: impossible to express the entire circle. However, it is doable in two parts: $\mathbf{c} = \mathbf{0}$, $r = 1$ where $y = \pm\sqrt{1 - x^2}$, $x \in [-1, 1]$
 - ▶ Parametric: $\mathbf{p}(t) = r \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \mathbf{c}$, where $t \in [0, 2\pi)$



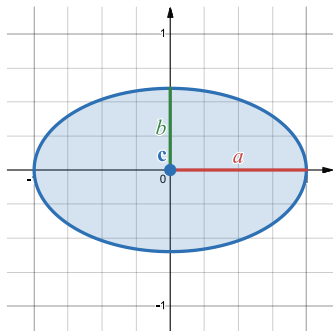
Ellipse

- ▶ The ellipse with center point $\mathbf{c} \in \mathbb{E}^2$, major axis parallel to the x axis, major axis $2a$ and minor axis $2b$ is:

- ▶ Implicit: $\frac{(x-c_x)^2}{a^2} + \frac{(y-c_y)^2}{b^2} = 1$

- ▶ Explicit form has the same problem as the circle (see above)

- ▶ Parametric: $\mathbf{p}(t) = \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix} + \mathbf{c}$,
where $t \in [0, 2\pi)$



Ellipse

- ▶ But what if, we don't want our axes to be parallel with x, y axes
 - ▶ Implicit: seems kind of elaborate (it's not but), we don't need it for now...
 - ▶ Parametric: Change the basis! If the new axes \mathbf{k}, \mathbf{l} , then $\mathbf{p}(t) = a \cos t \cdot \mathbf{k} + b \sin t \cdot \mathbf{l} + \mathbf{c}$, where $t \in [0, 2\pi)$

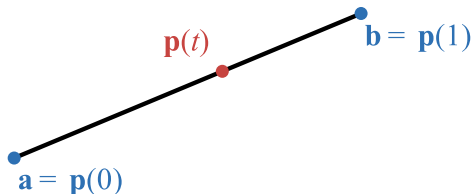
Segment

- ▶ Let points $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$. The parametric equation of the line going through two points:

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b},$$

where $t \in \mathbb{R}$.

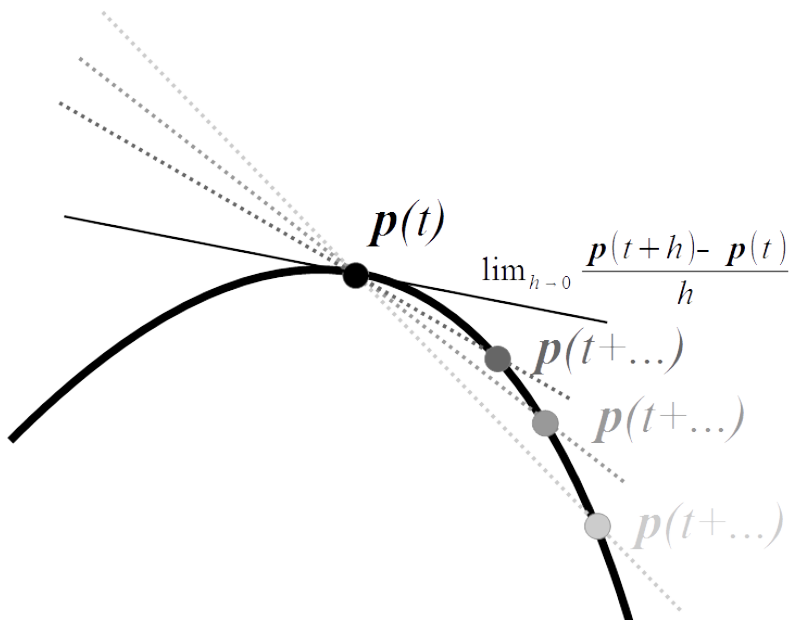
- ▶ If $t \in [0, 1]$, then the above gives the line segment connecting \mathbf{a}, \mathbf{b} .



Parametric form of curves

- ▶ Derivatives: $\mathbf{p}^{(i)}(t) = \begin{bmatrix} x^{(i)}(t) \\ y^{(i)}(t) \end{bmatrix}$, $t \in [\dots]$, $i = 0, 1, 2, \dots$
- ▶ If we consider the curve as the trajectory of a moving point, then the first derivative can be considered the velocity, the second the acceleration, etc.

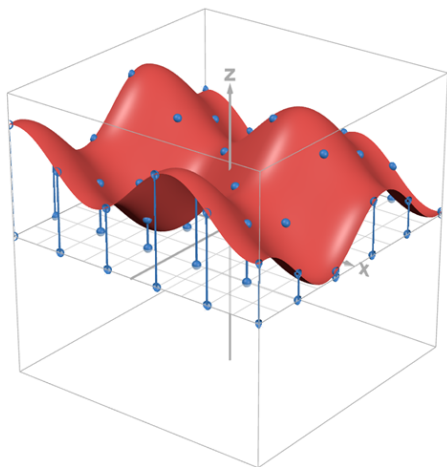
Tangent of the curve



Describing surfaces

- ▶ Explicit: $z = f(x, y) \rightarrow \{(x, y, f(x, y)) \mid (x, y) \in \mathcal{D}_f \subset \mathbb{R}^2\}$
- ▶ Implicit: $f(x, y, z) = 0 \rightarrow \{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^3 \mid f(\mathbf{x}) = 0\}$
- ▶ Parametric: $\mathbf{p}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}, (u, v) \in [a, b] \times [c, d]$
 $\rightarrow \{\mathbf{p}(u, v) \mid (u, v) \in \mathcal{D}_p \subset \mathbb{R}^2\}$
- ▶ How do we draw them?
 - ▶ explicit & parametric: generate triangles on the surface (see Incremental synthesis in Lectures 7-9.)...
 - ▶ implicit: we intersect the surface with rays (see Raycasting in Lecture 5.)...

Explicit surfaces



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

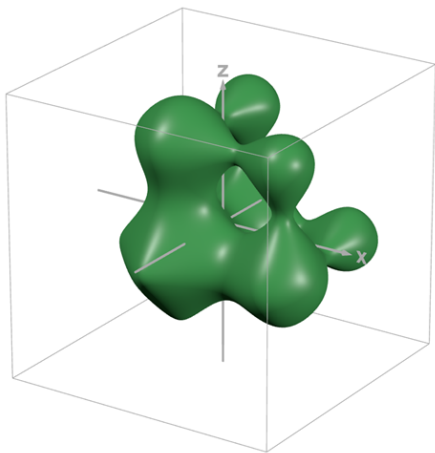
$$z = f(x, y), (x, y) \in \mathbb{R}^2$$

Set of points:

$$\{(x, y, f(x, y)) \mid (x, y) \in \mathcal{D}_f \subset \mathbb{R}^2\}$$

Example: $f(x, y) = \sin(x) + \sin(y) + 2$

Implicit surfaces



$$f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$\mathbf{x} = (x, y, z) \in \mathbb{R}^3$$

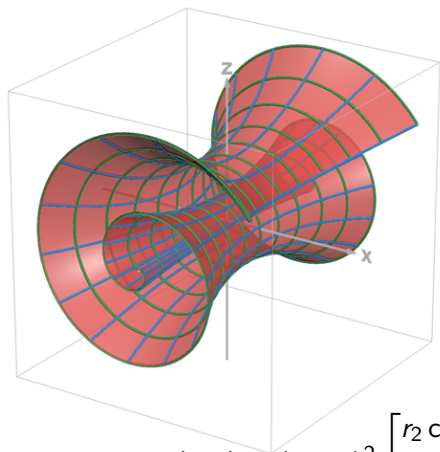
$$f(x, y, z) = 0$$

Set of points:

$$\{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^3 \mid f(\mathbf{x}) = 0\}$$

Example: $f(x, y, z) = x^2 + y^2 + z^2 + \sin 4x + \sin 4y + \sin 4z - 0.66$

Parametric surfaces



$$\mathbf{p} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\mathbf{p}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix},$$

$$(u, v) \in [a, b] \times [c, d]$$

Set of points:

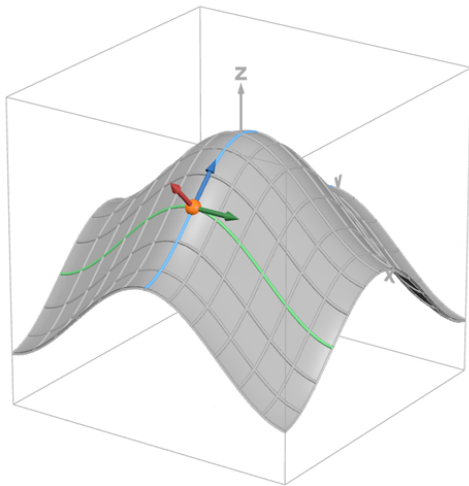
$$\{\mathbf{p}(u, v) \mid (u, v) \in \mathcal{D}_{\mathbf{p}} \subset \mathbb{R}^2\}$$

Example: $\mathbf{p}(u, v) = (1 - v)^2 \begin{bmatrix} r_2 \cos(r_1 u) u \\ -4 \\ r_2 \sin(r_1 u) u \end{bmatrix} + v^2 \begin{bmatrix} r_2 \cos(r_1 u) u \\ 4 \\ r_2 \sin(r_1 u) u \end{bmatrix}$

Surface normal of surfaces

- ▶ The surface normal in a given surface point is the normal of the surface's tangent plane there
- ▶ If the surface is given in parametric form:
$$\mathbf{n}(u, v) = \partial_u \mathbf{p}(u, v) \times \partial_v \mathbf{p}(u, v)$$
- ▶ For a surface given in implicit form $\mathbf{n}(x, y, z) = \nabla f$, where
$$\nabla f = [f_x, f_y, f_z]^T$$
- ▶ Describes which direction the surface is locally facing towards.
- ▶ Which will help us in computing light reflections on the surface, and shading the surface (later).

Surface normal of parametric surfaces



Sphere

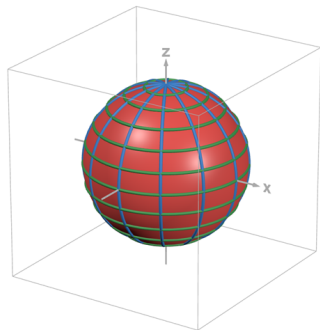
- ▶ Implicit:

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 = r^2$$

- ▶ Parametric:

$$\mathbf{p}(u, v) = r \begin{bmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{bmatrix} + \mathbf{c},$$

$$(u, v) \in [0, 2\pi) \times [0, \pi]$$



Ellipsoid

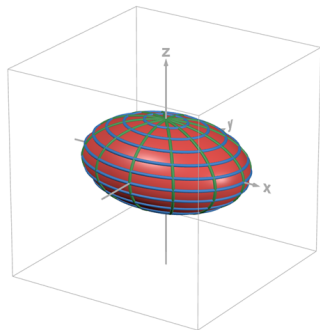
- Implicit:

$$\frac{(x-c_x)^2}{a^2} + \frac{(y-c_y)^2}{b^2} + \frac{(z-c_z)^2}{c^2} = 1$$

- Parametric:

$$\mathbf{p}(u, v) = \begin{bmatrix} a \cos u \sin v \\ b \sin u \sin v \\ c \cos v \end{bmatrix} + \mathbf{c},$$

$$(u, v) \in [0, 2\pi) \times [0, \pi]$$

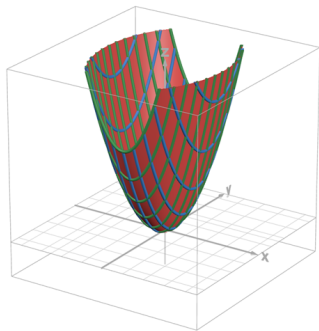


Simple paraboloid

► Parametric:

$$\mathbf{p}(u, v) = \begin{bmatrix} u \\ v \\ au^2 + bv^2 \end{bmatrix} + \mathbf{c},$$

$$(u, v) \in \mathbb{R}^2$$



A word of caution

- ▶ Most mathematical formulae treat z axis as the up direction
- ▶ This holds for the equations shown previously
- ▶ However, in computer graphics often y axis is up!

Notations

- ▶ \mathbf{l} is the vector toward the light „emitting” point, then the direction of incidence is $\mathbf{v} = -\mathbf{l}$
- ▶ \mathbf{n} surface normal
- ▶ $\mathbf{v}, \mathbf{l}, \mathbf{n}$ are unit vectors
- ▶ θ' is the angle between \mathbf{l} and \mathbf{n}

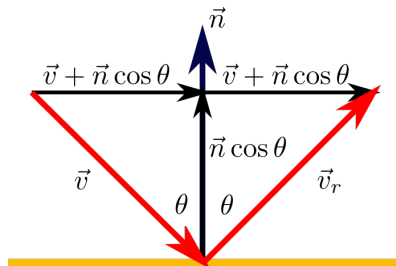
Ideal reflection

Law of reflection

The direction of incidence ($-\mathbf{l}$), the surface normal (\mathbf{n}), and the exit direction (\mathbf{r}) are in the same plane, and the angle of incidence (θ') is the same with the reflection angle (θ).

Direction of reflection

- ▶ In the general case, from an incident vector \mathbf{v} the reflection or specular direction:
- ▶ $\mathbf{v}_r = \mathbf{v} - 2\mathbf{n}(\mathbf{n} \cdot \mathbf{v})$
- ▶ Since $\cos \theta = -\mathbf{n} \cdot \mathbf{v}$, and \mathbf{n} , \mathbf{v} are unit vectors.



Ideal refraction

Snell–Descartes law

The incidence direction ($-\mathbf{l}$), the surface normal (\mathbf{n}), and the refraction direction (\mathbf{t}) are in the same plane, and $\eta = \frac{\sin \theta'}{\sin \theta}$, where η is the relative refractive index of the materials.

Relative refractive index

- ▶ $\eta = \frac{n_2}{n_1}$
- ▶ n_1 is the absolute refractive index of the medium the light is *leaving*.
- ▶ n_2 is the abs. refr. index of the medium the light is *entering*.

Absolute refractive indices

- ▶ Vacuum 1.0
- ▶ Air 1.0003
- ▶ Water 1.3333
- ▶ Glass 1.5
- ▶ Diamond 2.417

Direction of refraction

- ▶ Snell–Descartes law:

$$\eta = \frac{\sin \alpha}{\sin \beta}$$

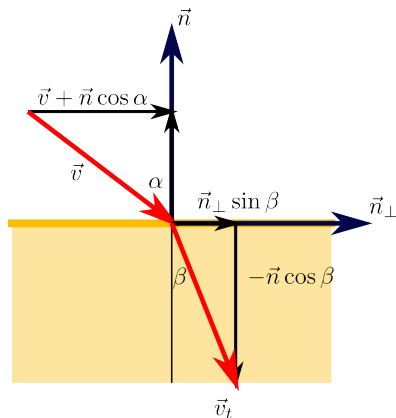
- ▶ $\mathbf{v}_t = \mathbf{n}_\perp \sin \beta - \mathbf{n} \cos \beta$

- ▶ $\mathbf{n}_\perp = \frac{\mathbf{v} + \mathbf{n} \cos \alpha}{\sin \alpha}$

- ▶ $\mathbf{v}_t = \frac{\mathbf{v}}{\eta} + \mathbf{n} \left(\frac{\cos \alpha}{\eta} - \cos \beta \right)$

- ▶ $\cos \beta = \sqrt{1 - \sin^2 \beta} = \sqrt{1 - \frac{\sin^2 \alpha}{\eta^2}}$

- ▶ $\mathbf{v}_t = \frac{\mathbf{v}}{\eta} + \mathbf{n} \left(\frac{\cos \alpha}{\eta} - \sqrt{1 - \frac{1 - \cos^2 \alpha}{\eta^2}} \right)$



Direction of refraction – remarks

- ▶ The refractive index, and as such the reflection direction is also dependent on the light's wavelength (see e.g. prisms)

$$\sqrt{1 - \frac{1 - \cos^2 \alpha}{\eta^2}}$$

- ▶ What if there is a negative number under the square root?
- ▶ This is called “total internal reflection”.
- ▶ Normally, some of the light is reflected, some of the light is refracted.
- ▶ When $\eta < 1$, i.e. $n_2 < n_1$, so the light is moving from the denser into the less dense medium, and α is such that this number is negative, all the light is reflected back and cannot leave the medium.
- ▶ But we have to be careful when implementing it...