# Computer Graphics

## Lecture 2: coordinate systems

Ágoston Sipos
siposagoston@inf.elte.hu

Eötvös Loránd University
Faculty of Informatics

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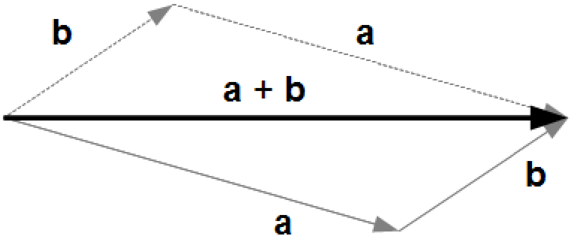
# Modelling

- How do we represent our virtual world?
  - How do we represent the points of our virtual space, how do we store it on the computer?
    $\rightarrow$ Coordinate systems
  - How do we represent the simple geometric building blocks (line, plane, triangle, etc.)?
    $\rightarrow$ Set of points
    $\rightarrow$ Description in different coordinate systems

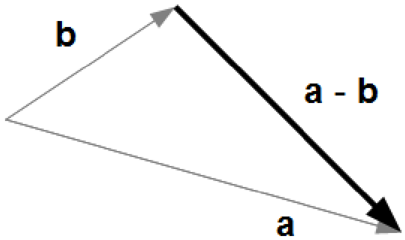
# Points, vectors

- Point: an element of the Euclidean plane/space that has no extent whatsoever.
  - geometrically: a position
- Vector:
  - algebraically: element of a vector space.
  - geometrically: a displacement that has *direction* and *magnitude*
  - additional operations defined on vectors: addition, subtraction, multiplication by a scalar, cross product (result is a vector), dot product (result is a scalar)

# Addition



# Subtraction



# Dot product

Let two vectors be, $\mathbf{a} = [a_x, a_y, a_z]$ and $\mathbf{b} = [b_x, b_y, b_z]$. Dot product is denoted by $\langle \mathbf{a}, \mathbf{b} \rangle$, i.e.

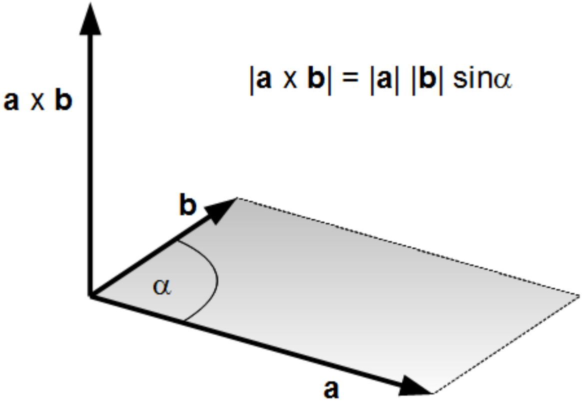
$$\langle \mathbf{a}, \mathbf{b} \rangle = a_x b_x + a_y b_y + a_z b_z.$$

This can also be expressed as

$$\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\alpha),$$

where $\alpha$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ vectors.

# Cross product (3D)




- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$: right-hand rule

# Cross product of vectors

Cross product as a determinant:

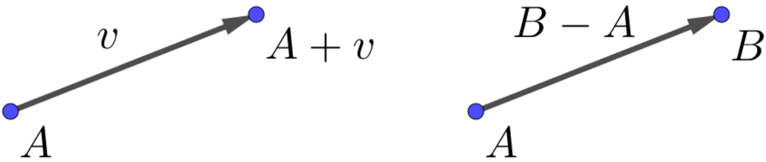
$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \mathbf{i} \cdot \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j} \cdot \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \cdot \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{bmatrix}$$

# Points, vectors

- Point and vector can be represented with coordinates of a chosen coordinate system. BUT: let's pay attention to the operations that can be performed!
- Operations between points and vectors:
  - point + vector = point  displaced point
  - point – point = vector  difference vector
  - point + point  not interpreted!



# Notation

- Points: $\mathbf{a} \in \mathbb{E}^2, \mathbf{b} \in \mathbb{E}^3$, ...
- Vectors: $\mathbf{v} \in \mathbb{R}^n, n = 2, 3, \ldots$
  - special: $[\mathbf{v}]_0 \in \mathbb{R}^n$ is a vector that is unit long i.e. $|\,[\mathbf{v}]_0\,| = ||\,[\mathbf{v}]_0\,||_2 = 1$.
- Lines: e, f, g, ...
- Planes: S, ...
- Matrices: $\mathbf{M} \in \mathbb{R}^{n \times m}$

# Coordinate system

- Uniquely represents point of space with n-tuples
  pl.: $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{E}^3$
- It allows storing points on the computer
- It allows the use of algebraic and analytical tools to solve geometric problems
- It may be easier to describe a problem in a coordinate system that fits it well

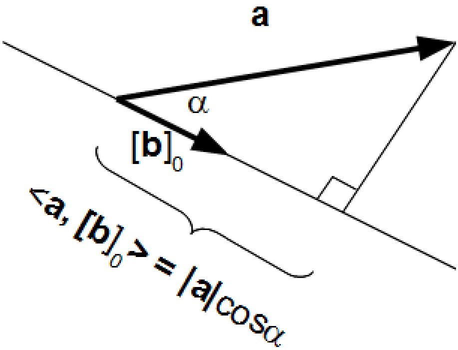
# Cartesian coordinate system

- Descartes, 1637.: *dissertation on the method* (Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences)
- Most of the time we come across this, this is the easiest and most common way of representing points

# Cartesian coordinate system

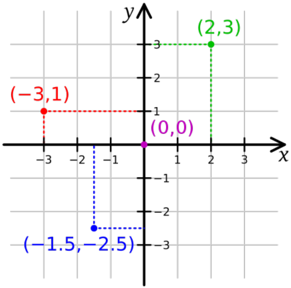
- For each finite point of the Euclidean space [plane] we assign an ordered, real number triplet $(x, y, z)$ [pair $(x, y)$]
- We define Cartesian coordinate system with a starting point (origin, $\mathbf{o}$) in the space, and an orthonormal system: three pairwise perpendicular unit vectors: $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ (these specify the direction of the x, y, z axes).
- Then the coordinates $x, y, z$ of a point $\mathbf{p}$ in row are the same as the signed orthogonal projections of the vector $\mathbf{p} - \mathbf{o}$ to the orthonormal basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.
- *Reminder*: the signed orthogonal projection of the vector $\mathbf{a}$ on unit vector $[\mathbf{b}]_0$ is $\langle \mathbf{a}, [\mathbf{b}]_0 \rangle = |\mathbf{a}| \cos \angle(\mathbf{a}, [\mathbf{b}]_0)$

# Signed orthogonal projection



# Geometric interpretation

- *More conceptually*: $\mathbf{p}(a, b, c)$ is the point from the origin which we get by moving $a$ units along the $x$ axis, then $b$ units along the $y$ axis, and finally $c$ units along $z$.



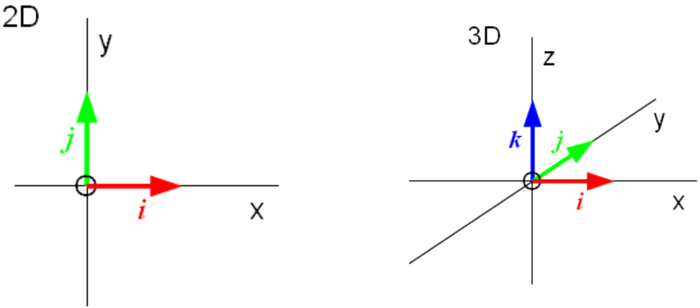
# Geometric interpretation

- According to the interpretation above, using the unit-long base vectors **i**, **j**, **k** pointing in the direction of the coordinate axes, $[a, b, c]^T$ the coordinates describe the following point:

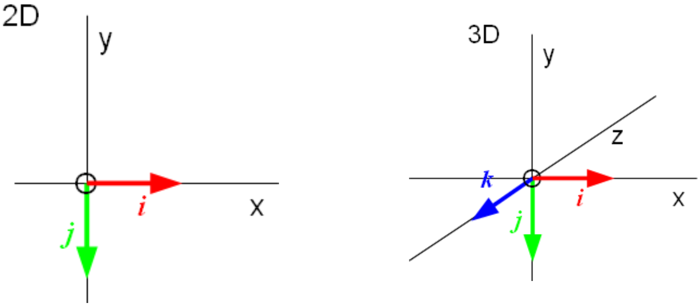
$$\mathbf{p} = \mathbf{o} + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$= \mathbf{o} + a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

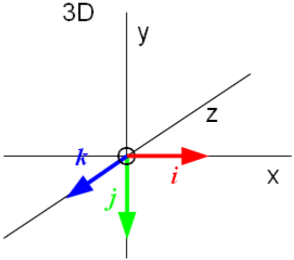
# Orientation – right-handed system

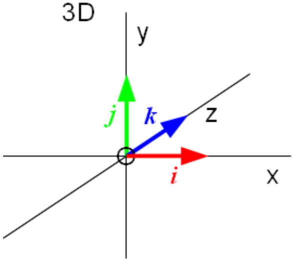


# Orientation – left-handed system



# Orientation – left-handed system



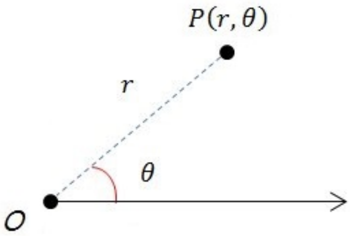



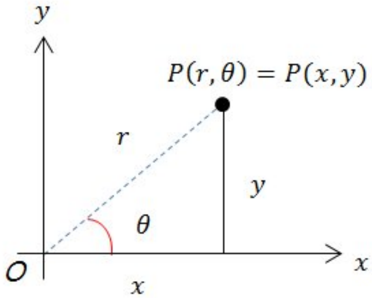
# Calculating distance

- The distance between two points **a** and **b** is $d$ if: $d^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$
- This is due to the Pythagorean theorem – as this constitutes a right-angled triangle
- $\rightarrow$ this opens up the possibility of describing new shapes (e.g. circle, sphere)
- Generally: $\mathbf{a}, \mathbf{b} \in \mathbb{E}^n : d = \sqrt{\langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$
- Length of a vector: $\mathbf{v} \in \mathbb{R}^n : ||\mathbf{v}||_2 = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\sum_{i=1}^{n} v_i^2}$

# Planar polar coordinate system

- Is defined by a starting point **o** (reference point) and a half-line starting from it (polar axis).
- The location of a point **p** is determined by two data: $(r, \theta)$
  - $r \geq 0$: is the distance between **p** and **o**
  - $\theta \in [0, 2\pi)$: is the angle between the polar axis and the half-line starting from **o** and going towards **p**



# Conversions

- Polar $\rightarrow$ Cartesian: $(r, \theta) \rightarrow (x, y)$
  - $x = r \cos\theta$
  - $y = r \sin\theta$
- Cartesian $\rightarrow$ Polar: $(x, y) \rightarrow (r, \theta)$
  - $r = \sqrt{x^2 + y^2}$
  -

$$\theta = \begin{cases} arctg(\frac{y}{x}), & x > 0 \wedge y \geq 0 \\ arctg(\frac{y}{x}) + 2\pi, & x > 0 \wedge y < 0 \\ arctg(\frac{y}{x}) + \pi, & x < 0 \\ \frac{\pi}{2}, & x = 0 \wedge y > 0 \\ \frac{3\pi}{2}, & x = 0 \wedge y < 0 \end{cases}$$
$$= atan2(y, x)$$

# Conversions

- The above is only true if the Cartesian origin and the polar reference point, respectively the Cartesian x-axis and the polar axis, are the same.
- What if $x = 0, y = 0$? In this case, with $r = 0$, we get back the origin with an arbitrary angle! The polar angle is not clear, we check whether $r = 0$ before trying to use the conversion formulas from above

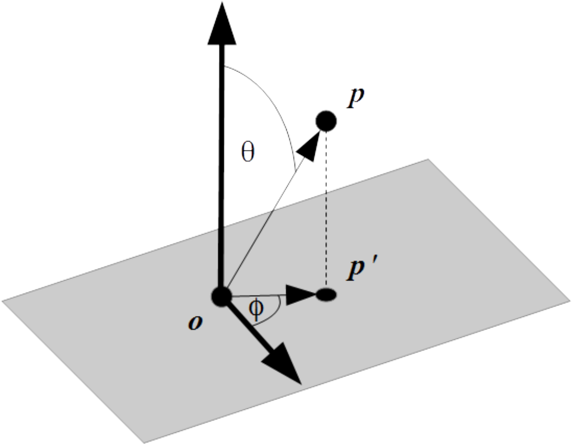
# Remark

- We usually use it when it fits well with the things we want to depict, e.g. circular motion
- Drawbacks: moving from one polar coordinate system (PCS) to another is expensive, calculating derivatives is expensive, ...

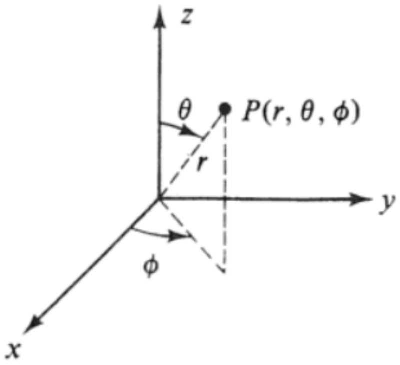
# Spherical coordinates

- Spherical coordinate system; defined by a base plane (and its PCS) and a perpendicular ”Z axis”
- A point **p** in space is represented by a number triplet: $(r, \theta, \phi)$
  - $(\varrho, \phi)$: polar coordinates of the projection of **p** onto the base plane
  - $\theta \in [0, \pi]$: the angle between Z axis and the half-line from **o** towards **p**
  - $r$: the distance between **p** and origin (if $r = 0$ then once again the two polar angles can be anything! This must be checked before conversions)

# Spherical coordinates



# Spherical coordinates



# Conversions

- Under conditions similar to the planar:
- Spherical → Cartesian: $(r, \theta, \phi) \rightarrow (x, y, z)$

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta$$

- Cartesian → Spherical: $(x, y, z) \rightarrow (r, \theta, \phi)$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = atan2(y, x), \qquad x^2 + y^2 \neq 0$$
$$\theta = \arccos \frac{z}{r}, \qquad r \neq 0$$

# Remark

- It is useful, for example, for identifying points on the earth's surface (but there $\theta \in [-\pi/2, \pi/2]$).
- The parametric representation of a sphere or ellipsoid also utilizes spherical coordinates

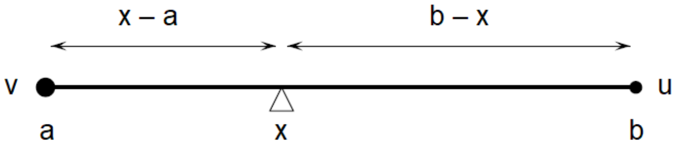
# Barycentric coordinates

- August Ferdinand Möbius [1827]
- Motivation: often only a specific, finite part of the space is interesting to us. We are looking for a representation more "balanced" than the Cartesian representation.

# Motivation: intervals


What $u, v$ weights should we place at the ends of the rod if we want the rod to stay in balance when elevated at the point denoted by a triangle?

# Motivation: intervals




- It does not tilt if $(x - a)v = (b - x)u$, where $x$ is the position of the triangle.
- Only the ratio of $u, v$ is bound by the above, let us further assume that $u + v = 1$
- Then the weights should be:

$$u = \frac{x - a}{b - a}, v = \frac{b - x}{b - a}$$

# Center of mass

- Mechanical analogy: center of mass for a point system
- Let us have 3 points and place weight $m_i \in \mathbb{R}$ in each $\mathbf{p}_i$ point. Then the center of mass is:

$$M = m_0 + m_1 + m_2, \qquad \mathbf{m} = \frac{m_0}{M} \cdot \mathbf{p}_0 + \frac{m_1}{M} \cdot \mathbf{p}_1 + \frac{m_2}{M} \cdot \mathbf{p}_2$$

- *Homogeneous* representation: multiplying the weights by a number $h \neq 0$ gives the same center of mass.

# Barycentric coordinates

- If $\mathbf{a}_0, ..., \mathbf{a}_n$ points in $\mathbb{E}^n$ span the space (that is, they do not fall into an $n-1$ dimensional subspace), then for any $\mathbf{x}$ point of the space we can find $\lambda_0, ..., \lambda_n$ real numbers that uniquely represent it such that

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{a}_i,$$

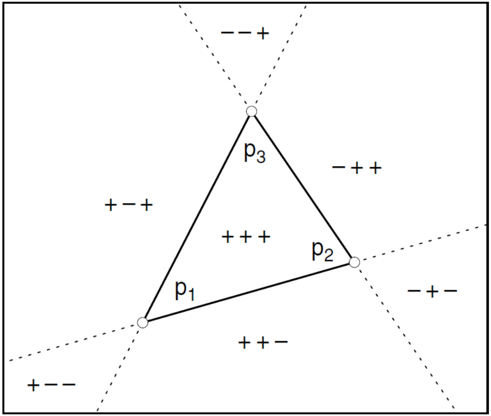
where $\lambda_i$ barycentric coordinates satisfy that,

$$\sum_{i=0}^{n} \lambda_i = 1.$$

# Remark

- In a plane, you need 3 affinely independent points (those that do not fall either in a straight line or a point), in space 4 affinely independent points
- If $\forall i\ \lambda_i \geq 0$, then we are talking about a convex combination and the result falls inside the convex hull of the points – in the plane on the triangle, in space inside the tetrahedron
- Affine transformations don't change barycentric coordinates (see later)

# Planar barycentric coordinate system




The $+/-$ signs denote the sign of the three barycentric coordinates for points falling into that part of the plane.

# Barycentric $\rightarrow$ Cartesian conversion

- Let $(u, v, w)$ be the barycentric coordinates of a point and $\mathbf{p}_1 = (x_1, y_1), \mathbf{p}_2 = (x_2, y_2), \mathbf{p}_3 = (x_3, y_3) \in \mathbb{E}^2$ affinely independent points.
- Then the Cartesian coordinates of the point $\mathbf{x}(x, y)$ represented by $(u, v, w)$ are $\mathbf{x} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, i.e.

$$x = ux_1 + vx_2 + wx_3$$
$$y = uy_1 + vy_2 + wy_3$$

# Planar barycentric coordinate system

- Let $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \begin{vmatrix} 1 & 1 & 1 \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{vmatrix}$, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^2$
- $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c})$ equals to twice the signed area of the triangle bound by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ points (positive if the vertices are given in counter-clockwise direction, otherwise it's negative)
- If we are in $\mathbb{E}^3$: $\Delta(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \langle (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}), \mathbf{n} \rangle$, where $\mathbf{n}$ is the unit long normal of the 3 points' plane.

# Cartesian → barycentric conversion

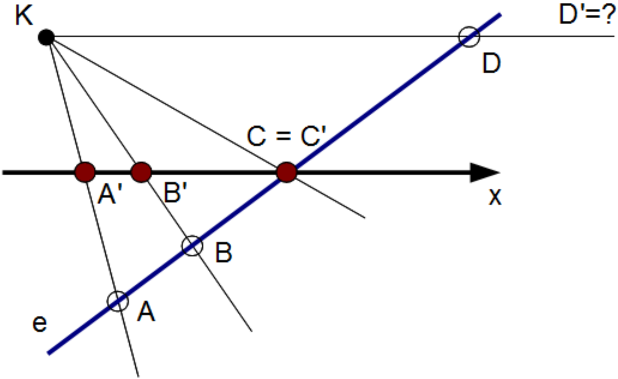
- Let $\mathbf{x} \in \mathbb{E}^2$ be a point, then its barycentric coordinates with $\mathbf{p}_1 = (x_1, y_1), \mathbf{p}_2 = (x_2, y_2), \mathbf{p}_3 = (x_3, y_3) \in \mathbb{E}^2$ affinely independent points are:

$$u = \frac{\Delta(\mathbf{x}, \mathbf{p}_2, \mathbf{p}_3)}{\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$$
$$v = \frac{\Delta(\mathbf{p}_1, \mathbf{x}, \mathbf{p}_3)}{\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$$
$$w = \frac{\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x})}{\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$$

[Demo]

# Motivation

- Project the points of a line $e$ onto the $x$ axis from point $\mathbf{k}$!



# Motivation

- The point $\mathbf{d}'$ is not on the Euclidean plane ($\mathbb{E}^2$), since the projection ray passing through $\mathbf{k}$ and $\mathbf{d}$ is parallel to the $x$ axis
- Idea: expand $\mathbb{E}^2$!
- $\rightarrow$ Let us consider the same orientation of the lines (their direction) as a point!
- This will be the *ideal point* of the line.

# Definition – ideal point

- Line = Line + 1 ideal point such that:
  - Parallel lines have the same ideal point ("they meet at infinity") – this is called the vanishing point e.g. in art
  - The ideal points of a plane lie on a line, this is the *ideal line of the plane*
  - The ideal line of parallel planes coincide
  - The ideal elements of space (points, lines) lie in a plane, this is the *ideal plane of the space*

# Definition and properties – homogeneous space

- Projective plane: the projective closure of $\mathbb{E}^2$, that is all the points of $\mathbb{E}^2$ and its ideal line
  - Two points determine a line in the projective plane
  - Two lines determine a point (!)
  - ...
- Projective space: the projective closure of $\mathbb{E}^3$, that is $\mathbb{E}^3$ plus its ideal plane
  - Three points determine a plane
  - Three distinct planes determine a point (!)
  - (HW: is it true that *any* (arbitrary) three planes define a point in projective space that is on all three planes? In what cases does it not?)
  - ...

# Homogeneous coordinates

- We assign a quadruple *homogeneous coordinates* for each point of the Euclidean space:

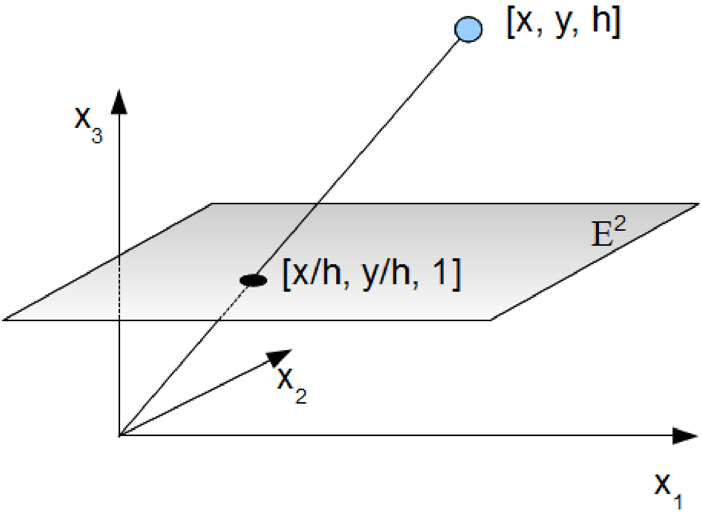
$$\begin{aligned}\mathbf{p}(x, y, z) &\rightarrow [x, y, z, 1] \\ &\approx h[x, y, z, 1] \\ &= [hx, hy, hz, h], \;\; h \neq 0\end{aligned}$$

- for every direction vector $\mathbf{v} = [x, y, z]^T$:

$$\begin{aligned}[x, y, z] &\rightarrow [x, y, z, 0] \\ &\approx h[x, y, z, 0] \\ &= [hx, hy, hz, 0], \;\; h \neq 0\end{aligned}$$

- Thus the homogeneous coordinates of a point or direction vector can be multiplied with any $h \neq 0$ number and the resulting coordinates represent the same point or direction vector.
  - When embedding $\mathbb{E}^n$ into $\mathbb{R}^{n+1}$, this means that all points on the projecting line mean the same euclidean point/vector.

# Embedding $\mathbb{E}^2$ into $\mathbb{R}^3$



# Converting back to Cartesian coordinate system

- What does $[x_1, x_2, x_3, x_4]$ represent in projective space?
  - If $x_4 \neq 0$, then we are talking about a point whose coordinates after homogeneous (or projective) division are:

$$[x_1, x_2, x_3, x_4] \approx \left[\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}, 1\right] = \mathbf{p}\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right)$$

  - If $x_4 = 0$, but $x_1^2 + x_2^2 + x_3^2 \neq 0$ (=not all zero), then it is an ideal point of a line whose orientation is same as $[x_1, x_2, x_3]$ vector.
  - If $x_i = 0$, $i = 1, 2, 3, 4$, then it is undefined.

# Notable homogeneous points

- Let $c \neq 0$ real number. Examples of notable points:
  - $[0, 0, 0, c]$ origin
  - $[c, 0, 0, 0]$ ideal point of x axis
  - $[0, c, 0, 0]$ ideal point of y axis
  - $[0, 0, c, 0]$ ideal point of z axis

# Properties

- On the projective plane, point and line, and in projective space, point and plane are dual concepts
- Note that some properties do not transfer from Euclidean space:
  - A point on a line does not divide the line into two parts! But: two different points do
  - A line does not divide a plane into two parts! But: two different lines do
  - Two points do not uniquely identify a segment! (The ideal point of the line "glues together" the two ends of the line)