Computer Graphics

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Motivation

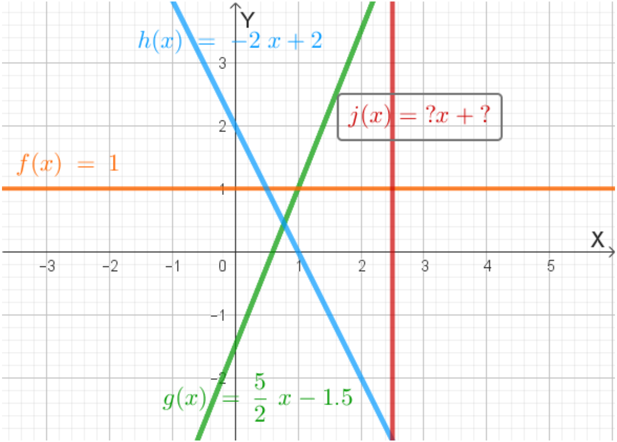
- We can now represent the points in different coordinate systems
- How can we describe simple things, such as a line or a plane?
- We primarily examine the above question in the Cartesian coordinate system

Curves and surfaces

- We can represent curves and surfaces (which include lines and planes too) as a set of points.
- How can we define these sets?
  - explicit: $y = f(x), x \in \mathbb{R} \quad \rightarrow \quad \{(x, f(x)) \mid x \in \mathcal{D}_f \subset \mathbb{R}\}$
    - what if we want to „reverse" it?
  - parametric: $\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in \mathbb{R}$
    $\rightarrow \quad \{\mathbf{p}(t) \mid t \in \mathcal{D}_\mathbf{p} \subset \mathbb{R}\}$
  - implicit: $f(x, y) = 0, (x, y) \in \mathbb{R}^2$
    $\rightarrow \quad \left\{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^2 \mid f(\mathbf{x}) = 0\right\}$

Line equation

- In high school: $y = mx + b$
- Problem: what about the vertical lines?



Normal vector equation of the line on the plane

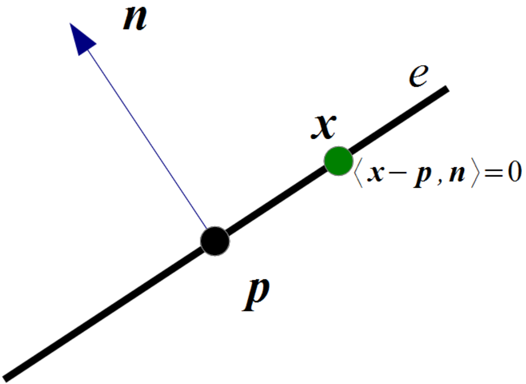
- Let $\mathbf{p}(p_x, p_y)$ be a point on the line and $\mathbf{n} = [n_x, n_y]^T \neq \mathbf{0}$ vector, a **normal** perpendicular to the direction of the line:
- All $\mathbf{x}(x, y)$ points on the line satisfy

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

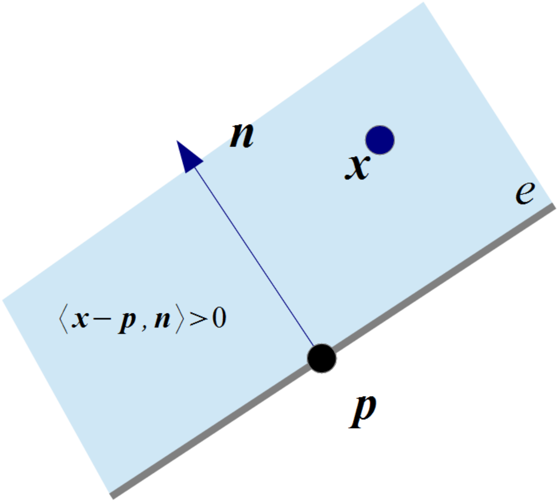
$$(x - p_x)n_x + (y - p_y)n_y = 0$$

  equation.
- $\langle \mathbf{x}' - \mathbf{p}, \mathbf{n} \rangle \; < 0$ and $\langle \mathbf{x}' - \mathbf{p}, \mathbf{n} \rangle \; > 0$ represents points on the two half-planes defined by our line.

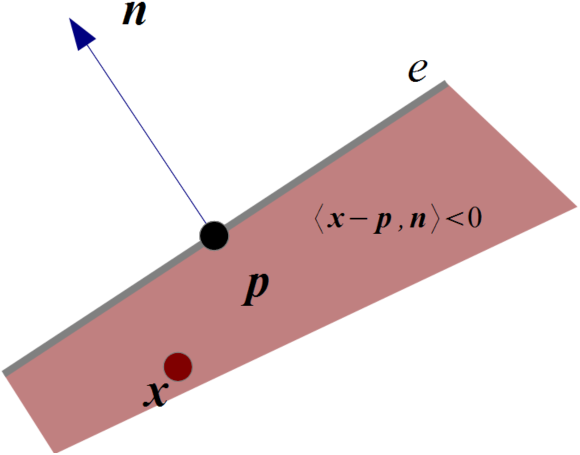
The two half-planes defined by line



The two half-planes defined by line



The two half-planes defined by line



The homogeneous implicit equation of the line on the plane

- The equation $ax + by + c = 0$ is the implicit equation of the line on the plane.
- In the previous representation, choosing $a = n_x$, $b = n_y$ and $c = -(p_x n_x + p_y n_y)$, $a^2 + b^2 \neq 0$ we get the implicit equation of the line going through $\mathbf{p}$, with $\mathbf{n}$ normal
- If $a^2 + b^2 = 1$, then this is the *Hesse normal form* – in this case the normal vector has a unit length

Homogeneous implicit equation with determinant

- Let $\mathbf{p}(p_x, p_y)$ and $\mathbf{q}(q_x, q_y)$ be two distinct points on the line then $\mathbf{x}(x, y)$ point belongs to the line if:

$$\begin{vmatrix} x & y & 1 \\ p_x & p_y & 1 \\ q_x & q_y & 1 \end{vmatrix} = 0$$

- Remark: the above determinant is the signed area of the triangle (twice the signed area) spanned by $\mathbf{x}(x, y)$, $\mathbf{p}(p_x, p_y)$, $\mathbf{q}(q_x, q_y)$, which $=0 \iff$ the points are in one line

Parametric equation of lines – with direction vector (2D, 3D)

- Let $\mathbf{p}(p_x, p_y, p_z)$ be a point on the line and $\mathbf{v} = [v_x, v_y, v_z]^T \neq \mathbf{0}$ a direction vector of the line (a vector parallel to the line)

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{p} + t\mathbf{v} = \begin{bmatrix} p_x + tv_x \\ p_y + tv_y \\ p_z + tv_z \end{bmatrix} \rightarrow \\
x(t) &= p_x + tv_x \\
y(t) &= p_y + tv_y \\
z(t) &= p_z + tv_z
\end{aligned}
$$

Parametric equation of lines – with two points (2D, 3D)

- Then $\mathbf{p}$ and $\mathbf{q}$ are the points of the line. We get the previous case by using $\mathbf{v} = \mathbf{q} - \mathbf{p}$:

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{p} + t\mathbf{v} \rightarrow \\
\mathbf{x}(t) &= (1-t)\mathbf{p} + t\mathbf{q} \rightarrow \\
x(t) &= (1-t)p_x + tq_x \\
y(t) &= (1-t)p_y + tq_y \\
z(t) &= (1-t)p_z + tq_z
\end{aligned}$$

*Homogeneous coordinate form

- We can represent a line of the extended (projective) plane with a real number triplet $\mathbf{e} = [e1, e2, e3]$, so called *line coordinate*, using which for every $\mathbf{x} = [x_1, x_2, x_3]^T$ point of the line
$$\mathbf{ex} = e_1 x_1 + e_2 x_2 + e_3 x_3 = 0$$
- The line coordinate of the ideal line including every $[x_1, x_2, 0]$ ideal point of the plane is $[0, 0, 1]$.

*Polar coordinate form

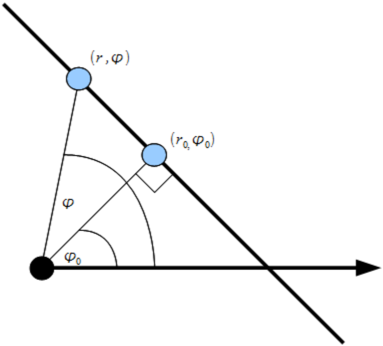
- The (implicit) equation of the half-line starting from the origin and making θ angle with the polar axis, in polar coordinate system:

$$\phi = \theta$$

- If our line does not pass through the origin, then let (r_0, ϕ_0) be the intersection of our line and a perpendicular line passing through the origin. Then, from the polar coordinates of our line, the radius can be written as a function of the polar angle in the following form:

$$r(\phi) = \frac{r_0}{\cos(\phi - \phi_0)}$$

*Polar coordinate form



Normal vector equation of the plane

- Let $\mathbf{p}(p_x, p_y, p_z)$ a point on the plane and $\mathbf{n} = [n_x, n_y, n_z]^T$ normal vector orthogonal to the plane, then for every $\mathbf{x}$ point of the plane:

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

- Half-spaces: $\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle \ < 0$, $\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle \ > 0$

Homogenous, implicit equation of the plane

- Implicit form of the plane $ax + by + cz + d = 0$
- From this with $a = n_x$, $b = n_y$, $c = n_z$ and $d = -n_x p_x - n_y p_y - n_z p_z$ we get the equation for the plane going through $\mathbf{p}$ point, $\mathbf{n}$ normal vector
- Hesse normal form here too, if $a^2 + b^2 + c^2 = 1$

Homogeneous implicit equation with determinant

- We can write the equation as a determinant, where it's zero for every X points of the plane spanned by $\mathbf{p}(p_x, p_y, p_z)$, $\mathbf{q}(q_x, q_y, q_z)$, $\mathbf{r}(r_x, r_y, r_z)$ points

$$\begin{vmatrix} x & y & z & 1 \\ p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \\ r_x & r_y & r_z & 1 \end{vmatrix} = 0$$

Parametric equation of the plane – using spanning vectors

- Let $\mathbf{p}$ point on the plane and $\mathbf{u}, \mathbf{v}$ spanning vectors of the plane (basis vectors):
$$\mathbf{x}(s, t) = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$
where $s, t \in \mathbb{R}$.

Parametric equation of the plane – using three points

- Three points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ not falling into one line define a plane, then we can get every finite $\mathbf{x}$ point of the plane
$$\mathbf{x}(s,t) = \mathbf{p} + s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p})$$
where $s, t \in \mathbb{R}$.
- We can get this from the former as well $\mathbf{u} = \mathbf{q} - \mathbf{p}$, $\mathbf{v} = \mathbf{r} - \mathbf{p}$
- This is a barycentric solution:
$$\mathbf{x}(s,t) = (1 - s - t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}$$
since $(1 - s - t) + s + t = 1$

*Homogenous coordinate form

- A plane in projective space can be represented with 4-tupple $\mathbf{s} = [s_1, s_2, s_3, s_4]$ a "plane-coordinate", which for every $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$ point of the plane

$$\mathbf{sx} = s_1 x_1 + s_2 x_2 + s_3 x_3 + s_4 x_4 = 0$$

*Notable planes in homogenous form

- $[0, 0, 0, c]$ ideal plane
- $[c, 0, 0, 0]$ the YZ plane
- $[0, c, 0, 0]$ the XZ plane
- $[0, 0, c, 0]$ the XY plane

Description of curves (2D)

- We can represent curves and surfaces (including lines and planes) as a set of points.
- How can we define these sets?
  - explicit: $y = f(x) \quad \rightarrow \quad \{(x, f(x)) \mid x \in \mathcal{D}_f \subset \mathbb{R}\}$
  - parametric: $\mathbf{p}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \rightarrow \quad \{\mathbf{p}(t) \mid t \in \mathcal{D}_\mathbf{p} \subset \mathbb{R}\}$
  - implicit: $f(x, y) = 0 \quad \rightarrow \quad \left\{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^2 \mid f(\mathbf{x}) = 0\right\}$
- But how do we draw them?
  - explicit & parametric: generate several points on the curve...
  - implicit: check pixels on screen if they are on the curve...

Transforming curves

How do we transform curves given in different representations?

- Explicit
  - Vertical translation and scaling: modifying the function value
    $\rightarrow y = a \cdot f(x) + b$
  - Horizontal translation and scaling: modifying the parameter
    $\rightarrow y = f(\frac{x}{c} - d)$
- Parametric: transforming the function value
  $\rightarrow \mathbf{A} \cdot \mathbf{p}(t)$
- Implicit: transforming the parameter with the inverse
  $\rightarrow f(\mathbf{A}^{-1} \cdot \mathbf{x}) = 0$

Parabola

- The parabola of focus point $(0, p)$ about y axis and crossing the origin can be written as
  - Implicit: $x^2 = 4py$
  - Explicit: $y = \frac{x^2}{4p}$, $x \in \mathbb{R}$
  - Parametric: $\mathbf{p}(t) = \begin{bmatrix} t \\ \frac{t^2}{4p} \end{bmatrix}$, $t \in \mathbb{R}$

Parabola

- What if we want to translate the parabola from origin into $\mathbf{c}$ point?
- In implicit and explicit formulation one has to work the coordinates of the translation (c_x, c_y) into the formulation (e.g. from the implicit we get $(x - c_x)^2 = 4p(y - c_y)$)
- In parametric form it is simply $\mathbf{p}(t) + \mathbf{c}$.

Circle

- A circle with $\mathbf{c} \in \mathbb{E}^2$ origin and r radius
  - Implicit form: $(x - c_x)^2 + (y - c_y)^2 = r^2$
  - Explicit: impossible to express the entire circle However, it is doable in two part: $\mathbf{c} = \mathbf{0}, r = 1$ where $y = \pm\sqrt{1 - x^2}$, $x \in [-1, 1]$
  - Parametric: $\mathbf{p}(t) = r \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \mathbf{c}$, where $t \in [0, 2\pi)$

Ellipse

- The ellipse with center point $\mathbf{c} \in \mathbb{E}^2$, major axis parallel to the x axis, major axis $2a$ and minor axis $2b$ is:
  - Implicit: $\frac{(x-c_x)^2}{a^2} + \frac{(y-c_y)^2}{b^2} = 1$
  - Explicit form has the same problem as the circle (see above)
  - Parametric: $\mathbf{p}(t) = \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix} + \mathbf{c}$, where $t \in [0, 2\pi)$

Ellipse

- But what if, we don't want our axes to be parallel with x, y axes
  - Implicit: seems kind of elaborate (it's not but), we don't need it for now...
  - Parametric: Change the base! If the new axes $\mathbf{k}, \mathbf{l}$, then $\mathbf{p}(t) = a \cos t \cdot \mathbf{k} + b \sin t \cdot \mathbf{l} + \mathbf{c}$, where $t \in [0, 2\pi)$

Segment

- Let points $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$. The parametric equation of the line going through two points:

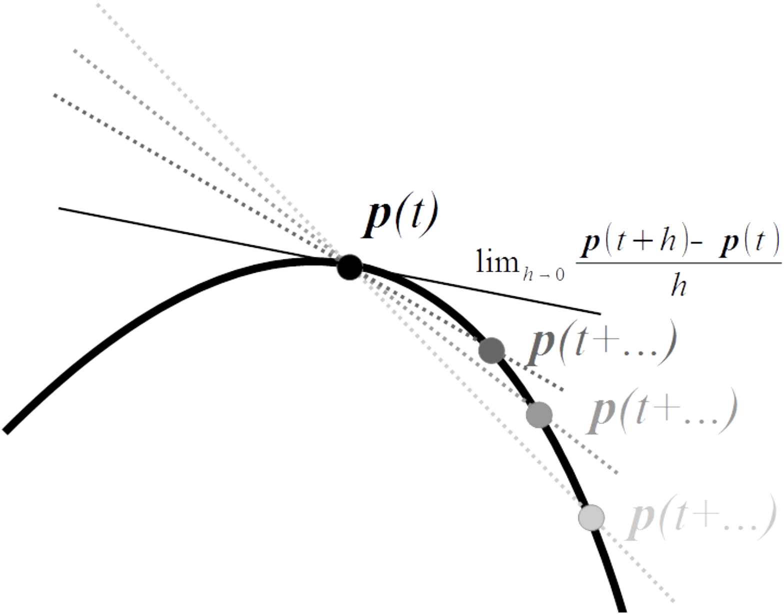
  $$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b},$$

  where $t \in \mathbb{R}$.
- If $t \in [0, 1]$, then the above gives the line segment connecting $\mathbf{a}, \mathbf{b}$.

Parametric form of curves

- Derivatives: $\mathbf{p}^{(i)}(t) = \begin{bmatrix} x^{(i)}(t) \\ y^{(i)}(t) \end{bmatrix}$, $t \in [...]$, $i = 0, 1, 2, ...$
- If we consider the curve as the trajectory of a moving point, then the first derivative can be considered the velocity, the second the acceleration, etc.

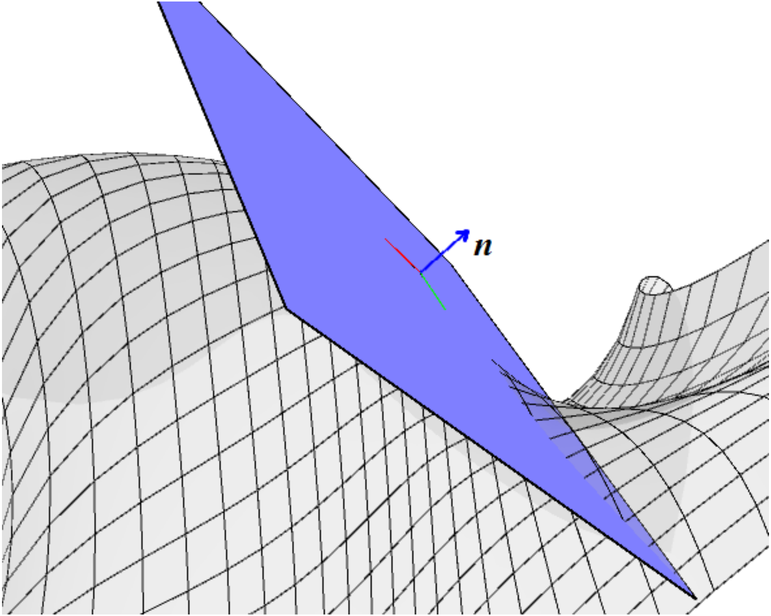
Tangent of the curve



Describing surfaces

- Explicit: $z = f(x, y) \quad \rightarrow \quad \left\{(x, y, f(x, y)) \mid (x, y) \in \mathcal{D}_f \subset \mathbb{R}^2\right\}$
- Implicit: $f(x, y, z) = 0 \quad \rightarrow \quad \left\{\mathbf{x} \in \mathcal{D}_f \subset \mathbb{R}^3 \mid f(\mathbf{x}) = 0\right\}$
- Parametric: $\mathbf{p}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$, $(u, v) \in [a, b] \times [c, d]$
  $\rightarrow \quad \left\{\mathbf{p}(u, v) \mid (u, v) \in \mathcal{D}_{\mathbf{p}} \subset \mathbb{R}^2\right\}$
- How do we draw them?
  - explicit & parametric: generate triangles on the surface (see Incremental sythesis in Lectures 7-9.)...
  - implicit: we intersect the surface with rays (see Raycasting in Lecture 5.)...

Surface normal of surfaces



Surface normal of surfaces

- Normal of the surface's tangent plane
- If the surface is given in parametric form:
  $\mathbf{n}(u, v) = \partial_u \mathbf{p}(u, v) \times \partial_v \mathbf{p}(u, v)$
- For a surface given in implicit form $\mathbf{n}(x, y, z) = \nabla f$, where
  $\nabla f = [f_x, f_y, f_z]^T$

Sphere

- Implicit: $(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 = r^2$
- Parametric:

$$\mathbf{p}(u, v) = r \begin{bmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{bmatrix} + \mathbf{c},$$

$(u, v) \in [0, 2\pi) \times [0, \pi]$

Ellipsoid

- Implicit: $\frac{(x-c_x)^2}{a^2} + \frac{(y-c_y)^2}{b^2} + \frac{(z-c_z)^2}{c^2} = 1$
- Parametric: $\mathbf{p}(u, v) = \begin{bmatrix} a \cos u \sin v \\ b \sin u \sin v \\ c \cos v \end{bmatrix} + \mathbf{c}$,

  $(u, v) \in [0, 2\pi) \times [0, \pi]$

Simple paraboloid

- Parametric: $\mathbf{p}(u,v) = \begin{bmatrix} u \\ v \\ au^2 + bv^2 \end{bmatrix} + \mathbf{c},\ (u,v) \in \mathbb{R}^2$

A word of caution

- Most mathematical formulae treats z axis as the up direction
- This holds for the equations shown previously
- However, in computer graphics often y axis is up!

Notations

- $\mathbf{l}$ is the vector toward the light „emitting" point, then the direction of incidence is $\mathbf{v} = -\mathbf{l}$
- $\mathbf{n}$ surface normal
- $\mathbf{v}, \mathbf{l}, \mathbf{n}$ are unit vectors
- θ' is the angle between $\mathbf{l}$ and $\mathbf{n}$

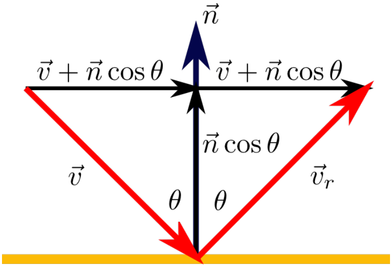
Ideal reflection

Law of reflection

The direction of incidence ($-\mathbf{l}$), the surface normal ($\mathbf{n}$), and the exit direction ($\mathbf{r}$) are in the same plane, and the angle of incidence (θ') is the same with the reflection angle (θ).

Direction of reflection

- In the general case, from an incident vector **v** the reflection or specular direction:
- $\mathbf{v}_r = \mathbf{v} - 2\mathbf{n}(\mathbf{n} \cdot \mathbf{v})$
- Since $\cos\theta = -\mathbf{n} \cdot \mathbf{v}$, and **n**, **v** are unit vectors.



Ideal refraction

Snell–Descartes law

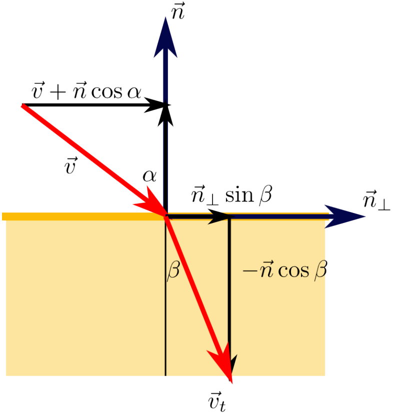
The incidence direction $(-\mathbf{l})$, the surface normal $(\mathbf{n})$, and the refraction direction $(\mathbf{t})$ are in the same plane, and $\eta = \frac{\sin\theta'}{\sin\theta}$, where η is the relative refractive index of the materials.

Some refractive indices

- Vacuum 1.0
- Air 1.0003
- Water 1.3333
- Glass 1.5
- Diamond 2.417

Direction of refraction

- Snell–Descartes law: $\eta = \frac{\sin\alpha}{\sin\beta}$
- $\mathbf{v}_t = \mathbf{n}_\perp \sin\beta - \mathbf{n}\cos\beta$
- $\mathbf{n}_\perp = \frac{\mathbf{v}+\mathbf{n}\cos\alpha}{\sin\alpha}$
- $\mathbf{v}_t = \frac{\mathbf{v}}{\eta} + \mathbf{n}\left(\frac{\cos\alpha}{\eta} - \cos\beta\right)$
- $\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{\sin^2\alpha}{\eta^2}}$



- $$\mathbf{v}_t = \frac{\mathbf{v}}{\eta} + \mathbf{n}\left(\frac{\cos\alpha}{\eta} - \sqrt{1-\frac{1-\cos^2\alpha}{\eta^2}}\right)$$